Chapter 1: Philosophy of Mathematics:

A Historical Introduction

1.0 Introduction

Mathematics presents itself as a science in the general sense in which history is a science, namely as sector in the quest for truth. Historians aim at establishing the truth about what was done by and what happened to human beings in the past.[12] The history of mathematics is primarily an investigation into the origin of discoveries in mathematics, the standard mathematical methods and notations of the past. In this chapter, first we make a brief survey of the history of mathematics in view of placing Gödel's Theorems within the historical trajectory of mathematics. Next we present contemporary developments in the philosophy of mathematics as a platform to delineate the relationship between mathematics and logic in general and also to expose the philosophical implications of Gödel's Incompleteness Theorem in particular.

1.1 Historical Phases in the Development of Mathematics

The most ancient mathematical texts available are *Plimpton 322* (Babylonian mathematics c. 1900 BCE), the *Moscow Mathematical Papyrus* (Egyptian mathematics c. 1850 BC), the *Rhind Mathematical Papyrus* (Egyptian mathematics c.

[12] Refer Michel Dummett, "What is Mathematics About?" in Alexander George (ed.), *Mathematics and Mind*, Oxford University Press, Oxford, 1994, 11-26.

1650 BC), and the *Shulba Sutras* (Indian mathematics c. 800 BC).[13] All these texts concern the so-called Pythagorean theorem, which seems to be the most ancient and widespread mathematical development after basic arithmetic and geometry. Egyptian and Babylonian mathematics were then further developed in Greek and Hellenistic mathematics, which is generally considered to be very important for greatly expanding both the method and the subject matter of mathematics. The mathematics developed in these ancient civilizations were further developed and greatly expanded in Islamic mathematics. Many Greek and Arabic texts on mathematics were then translated into Latin in medieval Europe and further developed there. One striking feature of the history of ancient and medieval mathematics is that bursts of mathematical development were often followed by centuries of stagnation. However the twentieth century witnessed tremendous development in mathematical sciences. [14]

1.2 Early Mathematics

In this section we deal with the ancient Indian mathematics, Babylonian Mathematics, Egyptian Mathematics, Vedic Mathematics, Greek and Hellenistic Mathematics, Classical Chinese Mathematics, Classical Indian Mathematics, Islamic mathematics and Modern Mathematics comprising of sixteenth to twenty first century.

1.2.1 Ancient Indian Mathematics (c. 3000-2600 BCE)

Long before the earliest written records, there are evidences that indicate some knowledge of elementary mathematics and of time measurement. The earliest known

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[13] Refer "Early History of Mathematics," in *http://www.globalshiksha.com/content/the-early-history-of-mathematics*, accessed on 12-11-2010.

[14] Refer "Early History of Mathematics," in *http://www.globalshiksha.com/content/the-early-history-of-mathematics*, accessed on 12-11-2010.

mathematics in ancient India dates from 3000-2600 BCE in the Indus Valley Civilization (*Harappan* civilization) of North India and Pakistan. [15] This civilization developed a system of uniform weights and measures that used the decimal system, a surprisingly advanced brick technology which utilized ratios, streets laid out in perfect right angles, and a number of geometrical shapes and designs, including cuboids, barrels, cones, cylinders, and drawings of concentric and intersecting circles and triangles. Mathematical instruments included an instrument for measuring the positions of stars for navigational purposes. The Indus script has not yet been deciphered; hence very little is known about the written forms of *Harappan* mathematics. Archeological evidence has led some to suspect that this civilization used a base 8 numeral system and had a value of π, the ratio of the length of the circumference of the circle to its diameter. [16]

1.2.2 Babylonian Mathematics (c. 1800-500 BCE)

Babylonian mathematics refers to any mathematics of the people of Mesopotamia (modern Iraq) from the days of the early Sumerians until the beginning of the Hellenistic period. It is named Babylonian mathematics due to the central role of Babylon as a place of study, which ceased to exist during the Hellenistic period. From this point, Babylonian mathematics merged with Greek and Egyptian mathematics to give rise to Hellenistic mathematics. Later, under the Arab Empire, Iraq/Mesopotamia, especially Baghdad, once again became an important center of

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[15] Refer Alexander Thom and Archie Thom, “The Metrology and Geometry of Megalithic Man,” in C.L.N. Ruggles (ed.), *Records in Stone: Papers in Memory of Alexander Thom*, Cambridge University Press, Cambridge, 1988, 132-151.

[16] Refer “Early History of Mathematics,” in *http://www.globalshiksha.com/content/the-early-history-of-mathematics*, accessed on 12-11-2010.

study for Islamic mathematics. Babylonian mathematics is known for clay tablets, written in Cuneiform script, unearthed since the 1850s.[17] Babylonian mathematics was written using a hexadecimal (base-60) numeral system. From this we derive the modern day usage of 60 seconds in a minute, 60 minutes in an hour, and 360 (60 x 6) degrees in a circle.

1.2.3 Egyptian Mathematics (c. 3000 - 300 BCE)

Egyptian mathematics refers to mathematics written in the Egyptian language. From the Hellenistic period, Greek replaced Egyptian as the written language of Egyptian scholars, and from this point Egyptian mathematics merged with Greek and Babylonian mathematics to give rise to Hellenistic mathematics. Mathematical study in Egypt later continued under the Arab Empire as part of Islamic mathematics, when Arabic became the written language of Egyptian scholars. The oldest mathematical text discovered so far is the Moscow papyrus, which is an Egyptian Middle Kingdom papyrus dated c. 2000-1800 BC.[18] Like many ancient mathematical texts, it consists of what are today called "word problems" or "story problems," which were apparently intended as entertainment. [19]

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[17] Asger Aaboe, *Episodes from the Early History of Mathematics*, Random House, New York, 1998, 30.

[18] *Ibid.*

[19] Refer "Early History of Mathematics," in *http://www.globalshiksha.com/content/the-early-history-of-mathematics*, accessed on 12-11-2010.

1.2.4 Vedic Mathematics (c. 900 BCE - 200 CE)[20]

It is believed that Vedic Mathematics originated from the Vedas manifest divine knowledge. Any knowledge derived from the Vedas is bound to have a touch of the divine bliss.[21] It began in the early Iron Age, with the *Shatapatha Brahmana* (c. 9th century BCE) and the *Sulba Sutras* (c. 800-500 BCE) which are geometry texts that used irrational numbers, prime numbers, the rule of three and cube roots.[22] They also computed the square root of 2 to five decimal places; solved linear equations and quadratic equations; developed Pythagorean triples algebraically and gave a statement and numerical proof of the Pythagorean theorem. Panini (c. 5 century BCE) formulated the grammar rules for Sanskrit. His notation was similar to modern mathematical notation and his grammar had the computing power equivalent to a Turing machine. Pingala (roughly 3rd-1st centuries BCE) in his treatise uses a device corresponding to a binary numeral system. His discussion of the combinatorics of meters, corresponds to the binomial theorem. Pingala's work also contains the basic ideas of Fibonacci numbers. Between 400 BCE and 200 CE, Jaina mathematicians began studying mathematics for the sole purpose of mathematics. They were the first to develop transfinite numbers, set theory, logarithms, fundamental laws of indices, cubic equations, quadratic equations, sequences and progressions, permutations and combinations, squaring and extracting square roots, and finite and infinite powers. The *Bakhshali Manuscript* written between 200 BC and 200 CE included quadratic

[20] Refer J.L. Banslal, "Leading mathematicians of Ancient India," in H.C. Khare (ed.), *Issues in Vedic Mathematics*, Rashtriya Veda Vidya Pratishthan in asscociation with Motilal Banarsidass Publishers, Delhi, 1991, 79. Hereafter it will be abbreviated as ***IVM*.** Refer "Vedic Mathematics," in *http://en.wikipedia.org/wiki/History_of_mathematics*, accessed on 20-10-2010.

[21] Narendra Puri, "An Overview of Vedic Mathematics," in ***IVM*,** 41.

[22] *Ibid.*

indeterminate equations, simultaneous equations, and the use of zero and negative numbers.

1.2.5 Greek and Hellenistic Mathematics (c. 550 BCE - 300 CE)

The foundations of mathematics and a great portion of its content, the first principles, the methods and the terminology, are Greek.[23] Greek mathematics refers to mathematics written in Greek. Greek mathematics was much more sophisticated than the mathematics that had been developed by earlier cultures. Greek mathematics of the period following Alexander the Great is sometimes called Hellenistic mathematics. All surviving records of pre-Greek mathematics show the use of inductive reasoning and Greek mathematicians, by contrast, used deductive reasoning. The Greeks used logic to derive conclusions from definitions and axioms.[24] Greek mathematics is thought to have begun with Thales (c. 624 - 546 BCE) and Pythagoras (c. 582- 507 BCE). Thales used geometry to solve problems such as calculating the height of pyramids and the distance of ships from the shore. Pythagoras is credited with the first proof of the Pythagorean Theorem, though the statement of the theorem has a long history. Aristotle (c. 384 - 322 BCE) first wrote down the laws of logic. Euclid (c. 300 BCE) is the earliest example of the format still used in mathematics today, definition, axiom, theorem, and proof.[25]

[23] Sir Thomas Heath, *A History of Greek Mathematics*, Vol. 1, Dover Publications, Inc., New York, 1981, v.

[24] Refer Martin Bernal, “On the Origins of Western Science,” in Michael H. Shank (ed.), *The Scientific Enterprise in Antiquity and the Middle Ages*, University of Chicago Pr., Chicago, 2000, 72-83.

[25] Howard Eves, *An Introduction to the History of Mathematics*, Saunders, 1990, 141.

1.2.6 Classical Chinese Mathematics (c. 500 BCE- 1300 CE)

The earliest mathematical works of China consists of binary tuples, trigrams and hexagrams for philosophical, mathematical, and/or mystical purposes. The binary tuples are composed of broken and solid lines, called *yin* 'female' and *yang* 'male' respectively. Chinese mathematics thrived at a time when European mathematics did not exist. [26] Developments first made in China and only much later known in the West, include negative numbers, the binomial theorem, matrix methods for solving systems of linear equations' and the Chinese remainder theorem. The Chinese used problems involving calculus, trigonometry, metrology, permutations, and once computed the possible amount of terrain space that could be used with specific battle formations, as well as the longest possible military campaign given the amount of food carriers could bring for themselves and soldiers. Even after European mathematics began to flourish during the Renaissance, European and Chinese mathematics were separate traditions, until the Jesuit missionaries such as Matteo Ricci carried mathematical ideas back and forth between the two cultures from the 16th to 18th centuries.

1.2.7 Classical Indian Mathematics (c. 400 BCE - 1600 CE)

The *Surya Siddhanta*[27] (c. 400) introduced the trigonometric functions of sine, cosine, and inverse sine, and laid down rules to determine the true motions of the luminaries, which conforms to their actual positions in the sky. Aryabhata in 499 CE produced the first trigonometric tables of sine, developed techniques and algorithms of algebra,

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[26] Refer "Early History of Mathematics," in *http://www.globalshiksha.com/content/the-early-history-of-mathematics*, accessed on 12-11-2010.

[27] The Surya Siddhanta is a *Siddhanta* treatise of Indian astronomy whose authorship is disputed.

infinitesimals, etc.[28] He also computed the value of π to the fourth decimal place as 3.1416. Madhava later in the 14th century computed the value of π to the eleventh decimal place as 3.14159265359. In the 7th century, Brahmagupta lucidly explained the use of zero as both a placeholder and decimal digit and explained the Hindu-Arabic numeral system. In the 12th century, Bhaskara first conceived differential calculus, along with the concepts of the derivative, differential coefficient and differentiation.[29] From the 14th century, Madhava and other Kerala School mathematicians developed the concepts fundamental to the overall development of calculus, including the mean value theorem, term by term integration, etc. In the 16th century, Jyeshtadeva consolidated many of the Kerala School's developments and theorems in the *Yuktibhasa*, the world's first differential calculus text, which also introduced concepts of integral calculus. Mathematical progress in India became stagnant from the late 16th century onwards due to subsequent political turmoil.

1.2.8 Islamic Mathematics (c. 800 - 1500 CE)

The Islamic Arab Empire established across the Middle East, Central Asia, North Africa, and in parts of India in the 8th century made significant contributions towards mathematics. In the 9th century, Muḥammad ibn Mūsā al-Ḵwārizmī wrote several important books on the Hindu-Arabic numerals and on methods for solving equations.[30] Al-Khwarizmi, often called the "father of algebra", for his fundamental

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[28] J. L. Banslal, "Leading mathematicians of Ancient India," 79-80.

[29] Bhaskaracharya (1114 CE), was renowned mathematician and astronemer of Vedic tradition, also author of Lilavati a comprehensive exposition of arithmetic, algebra and number theory. For details refer *Lilavati of Bhaskaracharya*, trans. K.S. Patwardhan, S.A. Naimpally, Shyma Lal Singh, Motilal Banarsidas Publishers, Pvt Ltd, Delhi, 2006, vi.

[30] Refer Victor J. Katz "Ideas of Calculus in Islam and India," *Mathematics Magazine* 68 (3), 1995, 174.

contributions to mathematics, gave an exhaustive explanation for the algebraic solution of quadratic equations with positive roots, and he was the first to teach algebra in an elementary form and for its own sake. He also introduced the fundamental method of "reduction". Al-Karaji around 1000 CE, introduced the theory of algebraic calculus.[31] In the late 11th century, Omar Khayyam laid the foundations for analytic geometry and non-Euclidean geometry. He was also the first to find the general geometric solution to cubic equations. The achievements of Muslim mathematicians includes algebra and algorithms, the development of spherical trigonometry, the addition of the decimal point notation to the Arabic numerals and also the discovery of all the modern trigonometric functions.[32]

1.3 Modern Mathematics (c. 1400-1600 CE)

Modern mathematics has extended its domain of inquiry to the rarefied abstractions of structures and the meticulous analysis of the foundations.[33] Beginning of Renaissance in Italy in the 16th century, new mathematical developments, interacting with new scientific discoveries, were made at an ever increasing pace, and this continues to the present day. In this section we deal with briefly on the mathematical developments from sixteenth century till twentieth century.

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[31] J. L. Berggren, "Innovation and Tradition in Sharaf al-Din al-Tusi's Muadalat," *Journal of the American Oriental Society* 110 (2), 1990, 304.

[32] Refer Early History of Mathematics in *http://www.globalshiksha.com/content/the-early-history-of-mathematics*, accessed on 12-11-2010.

[33] Piergiorgio Odifreddi, *The Mathematical Century: The 30 Greatest Problems of the Last 100 Years,* Universities Press, Hyderabad, 2005, 1.

1.3.1 Sixteenth and Seventeenth Century Mathematics[34]

In 16th century European mathematicians began to make advances without precedent anywhere in the world, so far as is known today. The first of these was the general solution of cubic equations, generally credited to Scipione del Ferro circa 1510, but first published by Johannes Petreius in Nuremberg which also included the solution of the general quadratic equation.[35] From this point on, mathematical developments came swiftly, contributing to and benefiting from contemporary advances in the physical sciences.[36]

The 17th century saw an unprecedented explosion of mathematical and scientific ideas across Europe. Galileo, an Italian, observed the moons of Jupiter in orbit about that planet, using a telescope based on a toy imported from Holland. Tycho Brahe, a Dane, had gathered an enormous quantity of mathematical data describing the positions of the planets in the sky. His student, Johannes Kepler, a German, succeeded in formulating mathematical laws of planetary motion. John Napier, in Scotland, was the first to investigate natural logarithms. The analytic geometry developed by René Descartes (1596-1650), a French mathematician and philosopher, allowed those orbits to be plotted on a graph, in Cartesian coordinates. Building on earlier work by many mathematicians, Isaac Newton, an Englishman, discovered the laws of physics explaining Kepler's Laws, and brought together the concepts now known as calculus. Independently, Gottfried Wilhelm Leibniz, in Germany, developed calculus and much

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[34] *Ibid.*

[35] Refer "History if Mathematics," in http://en.wikipedia.org/wiki/History_of_mathematics**,** accessed on 20-10-2010.

[36] Refer *Early History of Mathematics* in *http://www.globalshiksha.com/content/the-early-history-of-mathematics*, accessed on 12-11-2010.

of the calculus notation still in use today. Science and mathematics became an international endeavor, which would soon spread over the entire world. Pierre de Fermat and Blaise Pascal set the groundwork for the investigations of probability theory and the corresponding rules in their discussions over a game of gambling.[37]

1.3.2 Mathematics in the Eighteenth and Nineteenth Centuries[38]

The eighteenth century is known for the study of various number systems. This century also witnessed the development of continued fractions, decimals, the concept of the limit, and the study of a new constant "***e***" named after Leonhard Euler (1707 - 1783), who was very influential in the standardization of other mathematical terms and notations. He named the square root of minus 1 with the symbol ***i***. He also popularized the use of the Greek letter ***π*** to stand for the ratio of a circle's circumference to its diameter. Throughout the 19th century mathematics became increasingly abstract. It is in this century Carl Friedrich Gauss (1777 - 1855) gave the first satisfactory proofs of the fundamental theorem of algebra and of the quadratic reciprocity law. This century saw the development of hyperbolic geometry, elliptic geometry and the Riemannian geometry, which unifies and vastly generalizes the different types of geometry. During this century mathematicians dealt with abstract algebra, noncommutative algebra, Boolean algebra, set theory, mathematical logic and debate on the foundations of mathematics.[39]

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[37] *Ibid.*

[38] *Ibid.*

[39] Refer Early History of Mathematics in *http://www.globalshiksha.com/content/the-early-history-of-mathematics*, accessed on 12-11-2010.

1.3.3 Twentieth Century Mathematics

Developments in mathematics during the twentieth century comprised not only the creation of new branches and the extension of old ones but also the formulation of new approaches to previously established branches.[40] One of the more colorful figures in 20th century mathematics was Srinivasa Aiyangar Ramanujan (1887-1920) who conjectured or proved over 3000 theorems, including properties of highly composite numbers, the partition function and its asymptotics. He also made major discoveries in the areas of gamma functions, modular forms, divergent series, hypergeometric series and prime number theory. New areas of mathematics such as mathematical logic, topology, complexity theory, and game theory were emerged. At the same time, deep discoveries were made about the limitations to mathematics. In 1931, Kurt Gödel introduced his incompleteness theorem which explain that in any axiomatic mathematical system there are true statements that cannot be proved within the system. Hence mathematics cannot be reduced to mathematical logic, and David Hilbert's dream of making all of mathematics complete and consistent died.

1.4. Philosophy of Mathematics

Mathematics is a study which may be pursued in either of two opposite directions. One is more constructive, towards gradually increasing complexity of higher mathematics. And the other is more of analyzing to greater and greater abstractness and logical simplicity. The latter characterizes the philosophy of mathematics.[41] It is the branch of philosophy that studies the philosophical assumptions, foundations, and

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[40] J.W.A. Young (ed.), *Monographs on Topics of Modern Mathematics*, Dover Publications, Inc., New York, 1955, vii.

[41] Bertrand Russel, *Introduction to Mathematical Philosophy*, G.Allen and Unwin, London, 1920, 8.

implications of mathematics.[42] The recurrent themes include:

- What are the sources of mathematical subject matter?
- What is the ontological status of mathematical entities?
- What does it mean to refer to a mathematical object?
- What is the character of a mathematical proposition?
- What is the relation between logic and mathematics?
- What kinds of inquiry play a role in mathematics?
- What are the objectives of mathematical inquiry?
- What gives mathematics its hold on experience?
- What are the human traits behind mathematics?
- What is the source and nature of mathematical truth?
- What is the relationship between the abstract world of mathematics and the material universe?
- Is mathematics an absolute and universal language?

There are traditions of mathematical philosophy in both Western philosophy and Eastern philosophy. Western philosophies of mathematics go as far back as Plato, who studied the ontological status of mathematical objects, and Aristotle, who studied logic and issues related to infinity (actual versus potential).[43] Greek philosophy on mathematics was strongly influenced by their study of geometry. At one time, the Greeks held the opinion that 1 (one) was not a number, but rather a unit of arbitrary length. A number was defined as a multitude. Later in the recent centuries the focus

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[42] Donald M. Borchert (ed.), *Encyclopedia of Philosophy* (2nd Edition), Macmillan Reference, USA, 2006, 20. Hereafter it will be abbreviated as *EP*.

[43] Ted Honderich, *The Oxford Companion to Philosophy*, Oxford University Press, New York, 1995, 533-534.

shifted strongly to the relationship between mathematics and logic and this view dominated the philosophy of mathematics in the recent centuries especially in the 20th century.

1.4.1 Philosophy of Mathematics in the Twentieth Century

A perennial issue in the philosophy of mathematics concerns the relationship between logic and mathematics at their joint foundations. The philosophy of mathematics in the 20th century was characterized by a predominant interest in formal logic, set theory, and foundational issues.[44] It is a profound puzzle that on the one hand mathematical truths seem to have a compelling inevitability, but on the other hand the source of their "truthfulness" remains elusive. Investigations into this issue are known as the foundations of mathematics program.

At the start of the 20th century, philosophers of mathematics were already beginning to divide into various schools of thought about all these questions, broadly distinguished by their pictures of mathematical epistemology and ontology. Three schools, formalism, intuitionism, and logicism, emerged at this time, partly in response to the increasingly widespread worry that mathematics as it stood, and analysis in particular, did not live up to the standards of certainty and rigor that had been taken for granted. Each school addressed the issues that came to the fore at that time; either attempting to resolve them or claiming that mathematics is not entitled to its status as our most trusted knowledge.

Surprising and counterintuitive developments in formal logic and set theory early in the 20th century led to new questions concerning what was traditionally called the

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[44] Borchert, *EP*, 21.

foundations of mathematics. As the century unfolded, the initial focus of concern expanded to an open exploration of the fundamental axioms of mathematics, the axiomatic approach having been taken for granted since the time of Euclid around 300 BCE as the natural basis for mathematics. Notions of axiom, proposition and proof, as well as the notion of a proposition being true of a mathematical object (mathematical logic)), were formalized, allowing them to be treated mathematically. In mathematics, new and unexpected ideas had arisen and significant changes were coming. With Gödel numbering, propositions could be interpreted as referring to themselves or other propositions, enabling inquiry into the consistency of mathematical theories. This reflective critique in which the theory under review "becomes itself the object of a mathematical study" led to the study of *metamathematics*. Philosophy of mathematics today proceeds along several different lines of inquiry, by philosophers of mathematics, logicians, and mathematicians, and there are many schools of thought on the subject. The thoughts of a few significant schools are addressed below separately.

1.4.2 Contemporary Schools in Philosophy of Mathematics

In this section we deal with the schools of Mathematical Realism, Platonism, Logicism, Empiricism, the New Empiricism, Formalism, Intuitionalism, Fictionalism, Constructivism and Embodied Mind Theories to get clear picture of the philosophical developments in mathematics in the contemporary times.

1.4.2.1 Mathematical Realism

Mathematical Realism, like Realism in general, holds that mathematical entities exist independently of the human mind.[45] Thus humans do not invent mathematics, but

[45] Stewart Shapiro, *Philosophy of Mathematics: Structure and Ontology*, Oxford University Press, New

rather discover it, and any other intelligent beings in the universe would presumably do the same. In this point of view, there is really one sort of mathematics that can be discovered. Triangles, for example, are real entities, not the creations of the human mind.

Many working mathematicians have been mathematical realists; they see themselves as discoverers of naturally occurring objects. Example includes Kurt Gödel. Gödel believed in an objective mathematical reality that could be perceived in a manner analogous to sense perception. Certain principles (e.g., for any two objects, there is a collection of objects consisting of precisely those two objects) could be directly seen to be true, but some conjectures, like the continuum hypothesis, might prove undecidable just on the basis of such principles. Gödel suggested that quasi-empirical methodology could be used to provide sufficient evidence to be able to reasonably assume such a conjecture. Within realism, there are distinctions depending on what sort of existence one takes mathematical entities to have, and how we know about them.[46]

1.4.2.2 Platonism[47]

Platonism is the form of realism that suggests that mathematical entities are abstract, have no spatio-temporal or causal properties, and are eternal and unchanging. This is often claimed to be the view most people have of numbers. The term Platonism is used because such a view is seen to parallel Plato's belief in a "World of Ideas" that the everyday world can only imperfectly approximate of an unchanging, ultimate

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York, 1997, 4.

[46] Refer Penelope Maddy, *Realism in Mathematics*, Oxford University Press, Oxford, UK, 1990, 20ff.

[47] Borchert, *EP*, 35.

reality. Both Plato's ideas and Platonism have meaningful connections, because Plato's ideas were preceded and probably influenced by the hugely popular Pythagoreans of ancient Greece, who believed that the world was, quite literally, generated by numbers.[48]

The major problem of mathematical Platonism is this: precisely where and how do the mathematical entities exist, and how do we know about them? Is there a world, completely separate from our physical one, which is occupied by the mathematical entities? How can we gain access to this separate world and discover truths about the entities? It calls for the formulation of a theory that postulates all structures that exist mathematically also exist physically in their own universe.

1.4.2.3 Logicism

Logicism is the calim that mathematics is part of logic.[49] Logicists hold that mathematics can be known *a priori*, but suggest that our knowledge of mathematics is just part of our knowledge of logic in general, and is thus analytic, not requiring any special faculty of mathematical intuition. In this view, logic is the proper foundation of mathematics, and all mathematical statements are necessary logical truths.

Gottlob Frege was the founder of logicism. In his seminal *Basic Laws of Arithmetic* he built up arithmetic from a system of logic with a general principle of comprehension, a principle that he took to be acceptable as part of logic.[50] Rudolf

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[48] James Robert Brown, *Philosophy of Mathematics: An Introduction to the World of Proofs and Pictures*, Routledge, London and New York, 200, 9.

[49] Hartry Field, "Is Mathematical Knowledge Just Logical Knowledge," in W.D. Hart, *Philosophy of Mathematics*, Oxford University Press, New York, 235.

[50] Frege called this principle "Basic Law V" (for concepts *F* and *G*, the extension of *F* equals the extension of *G* if and only if for all objects *a*, *Fa* if and only if *Ga*). For details refer Michael Dummett,

Carnap (1931) presents the logicist thesis in two parts: (1) The *concepts* of mathematics can be derived from logical concepts through explicit definitions. (2) The *theorems* of mathematics can be derived from logical axioms through purely logical deduction. If mathematics is a part of logic, then questions about mathematical objects reduce to questions about logical objects. But what, one might ask, are the objects of logical concepts? In this sense, logicism can be seen as shifting questions about the philosophy of mathematics to questions about logic without fully answering them.[51]

1.4.2.4 Empiricism

Empiricism is a form of realism that denies that mathematics can be known a priori at all.[52] It says that we discover mathematical facts by empirical research, just like facts in any of the other sciences. It is not one of the classical three positions advocated in the early 20th century, but primarily arose in the middle of the century. However, an important early proponent of a view like this was John Stuart Mill. For him mathematical statements come out as uncertain, contingent truths, which can only be learnt by observing instances of two pairs coming together and forming a quartet.

Contemporary mathematical empiricism, formulated by Quine and Putnam, is primarily supported by the *indispensability argument*: mathematics is indispensable to all empirical sciences, and if we want to believe in the reality of the phenomena described by the sciences, we ought also believe in the reality of those entities

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Frege, Philosophy of Mathematics, Harvard University Press, Cambridge, MA., 1991. Also refer George Boolos, "The Consistency of Fege's Foundations of Arithmatic," in Hart, *Philosophy of Mathematics*, Oxford University Press, 185-202.

[51] Refer Hartry Field, "Is Mathematical Knowledge Just Logical Knowledge," in W.D. Hart, *Philosophy of Mathematics*, Oxford University Press, New York, 235-271.

[52]Borchert, *EP*, 217.

required for this description.[53] That is, since physics needs to talk about electrons to say why light bulbs behave as they do, then electrons must exist. Since physics needs to talk about numbers in offering any of its explanations, then numbers must exist. In keeping with Quine and Putnam's overall philosophies, this is a naturalistic argument. It argues for the existence of mathematical entities as the best explanation for experience, thus stripping mathematics of some of its distinctness from the other sciences. The most important criticism of empirical views of mathematics is approximately the same as that raised against Mill. If mathematics is just as empirical as the other sciences, then this suggests that its results are just as fallible as theirs, and just as contingent.

1.4.2.5 The New Empiricism[54]

A more recent empiricism returns to the principle of the English empiricists of the 18th and 19th centuries, in particular John Stuart Mill, who asserted that all knowledge comes to us from observation through the senses. This applies not only to matters of fact, but also to "relations of ideas." To this principle it adds a materialist concept: thoughts, ideas, all the processes of logic which organize, interpret and abstract observations, are physical phenomena which take place in real time and physical space: namely, in the brains of human beings. Abstract objects, such as mathematical objects, are ideas, which in turn exist as electrical and chemical states of the billions of neurons in the human brain.

This second concept is typical of the social constructivist approach, which holds that

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[53] For details refer W.V. quine, "Two Dogmas of Empiricism," in W.D. Hart, *Philosophy of Mathematics*, Oxford University Press, New York, 31-51. Also refer Hilary Putnam, "Mathematics Without Foundations," in Hart, Philosophy of Mathematics, 168-184.

[54] Borchert, *EP*, 221.

mathematics is produced by humans rather than being “discovered” from abstract, a priori truths. However, it differs sharply from the constructivist implication that humans arbitrarily and creatively construct mathematical principles that have no inherent truth but are merely social conventions agreed to by the society that invented them. On the contrary, the new empiricism insists that mathematics, although constructed by humans, follows rules and principles that are agreed on by all who participate in the process, with the result that everyone practicing mathematics comes up with the same answer - except in those areas where there is philosophical disagreement on the meaning of fundamental concepts. This agreement is a physical phenomenon, to be observed by other humans in the same way that physical phenomena like the motions of inanimate bodies or the chemical interaction of various elements is observed.

Kant argued that the structures of logic which organize, interpret and abstract observations were built into the human mind and were true and valid *a priori*. Mill, on the contrary, said that we believe them to be true because we have enough individual instances of their truth to generalize: in his words, “From instances we have observed, we feel warranted in concluding that what we found true in those instances holds in all similar ones, past, present and future, however numerous they may be.”[55]

For most mathematicians the empiricist principle that all knowledge comes from the senses contradicts a more basic principle: that mathematical propositions are true independent of the physical world. Everything about a mathematical proposition is

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[55] *A System of Logic Ratiocinative and Inductive*, The Collected Works of John Stuart Mill published by the University of Toronto Press in 1973, Book II, Chapter vi, Section 2, Toronto edition 1975, Vol.7, 254.

independent of what appears to be the physical world. It all takes place in the mind. And the mind operates on infallible principles of deductive logic. It is not influenced by exterior inputs from the physical world, distorted by having to pass through the tentative, contingent universe of the senses.

If this is true, then where do the senses come in? A large number of observers today consider it undeniable. It is admittedly disturbing, raising many questions about human values as well as contradicting most religious beliefs. But it is also liberating, relieving us of a vast library of puzzles and paradoxes that come with trying to conceptualize an immaterial world of the mind. And it reconciles the contradiction between our belief in the certainty of abstract deductions and the empiricist principle that knowledge comes from observation of individual instances. Indeed, the very principles of logical deduction are true because we observe that using them leads to true conclusions.

1.4.2.6 Formalism[56]

Formalism holds that mathematical statements may be thought of as statements about the consequences of certain string manipulation rules. Mathematical truths are not about numbers and sets and triangles and the like, in fact, they are not "about anything at all." Another version of formalism is often known as deductivism. In deductivism, the Pythagorean theorem is not an absolute truth, but a relative one: if you assign meaning to the strings in such a way that the rules of the game become true (ie, true statements are assigned to the axioms and the rules of inference are truth-preserving), then you have to accept the theorem, or, rather, the interpretation you have given it

[56] Refer Ted Honderich, *The Oxford Companion to Philosophy*, Oxford University Press, New York, 1995, 285-286.

must be a true statement. The same is held to be true for all other mathematical statements. Thus, formalism need not mean that mathematics is nothing more than a meaningless symbolic game. It is usually hoped that there exists some interpretation in which the rules of the game hold. But it does allow the working mathematician to continue in his or her work and leave such problems to the philosopher or scientist. Many formalists would say that in practice, the axiom systems to be studied will be suggested by the demands of science or other areas of mathematics.

A major early proponent of formalism was David Hilbert, whose program was intended to be a complete and consistent axiomatization of all of mathematics. (Consistent here means that no contradictions can be derived from the system.) Hilbert's goals of creating a system of mathematics that is both complete and consistent was dealt a fatal blow by the second of Gödel's theorems, which states that sufficiently expressive consistent axiom systems can never prove their own consistency. Since any such axiomatic system would contain the finitary arithmetic as a subsystem, Gödel's theorems implied that it would be impossible to prove the system's consistency relative to that (since it would then prove its own consistency, which Gödel had shown was impossible). Thus, in order to show that any axiomatic system of mathematics is in fact consistent, one needs to first assume the consistency of a system of mathematics that is in a sense stronger than the system to be proven consistent.

The main critique of formalism is that the actual mathematical ideas that occupy mathematicians are far removed from the string manipulation games mentioned above. Formalism is thus silent to the question of which axiom systems ought to be studied, as none is more meaningful than another from a formalistic point of view.

1.4.2.7 Intuitionism[57]

Mathematics for intuitionists is a systematic body of mathematical propositions verified in practice by rigorous mathematical proof. An intuitionist places epistemic limitations on the truth of mathematical propositions. A major force behind Intuitionism was L.E.J. Brouwer**,** who rejected the usefulness of formalized logic of any sort for mathematics.[58] According to Brouwer, in mathematics, intuitionism is a program of methodological reform whose motto is that "there are no non-experienced mathematical truths."[59] From this springboard, intuitionists seek to reconstruct what they consider to be the corrigible portion of mathematics in accordance with Kantian concepts of being, becoming, intuition, and knowledge. Brouwer, the founder of the movement, held that mathematical objects arise from the *a priori* forms of the volitions that inform the perception of empirical objects.

For Arend Heyting the intuitionistic logic is different from the classical Aristotelian logic.[60] The intuitionist logic does not contain the law of the excluded middle and therefore frowns upon proofs by contradiction. The axiom of choice is also rejected in most intuitionistic set theories, though in some versions it is accepted. However in intuitionism, the terms like "explicit construction" is not cleanly defined, and that has led to criticisms. Attempts have been made to use the concepts of Turing machine or computable function to fill this gap, leading to the claim that only questions regarding the behavior of finite algorithms are meaningful and should be investigated in mathematics. This has led to the study of the computable numbers, first introduced by

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[57]Borchert, *EP*, 39.

[58] L.E.J. Brouwer is one of the leading founders of intuitionism. Refer Dale Jacquette (ed.), *Philosophy of Mathematics: An Anthology*, , Blackwell Publishers, Oxford, 2002, 262.

[59] *Ibid.*

[60] Arend Heyting, Intuitionism: An Introduction, 3rd Edition, North Holland, Asterdam, 1971, 15.

Alan Turing.

1.4.2.8 Constructivism[61]

Like intuitionism, constructivism involves the regulative principle that only mathematical entities which can be explicitly constructed in a certain sense should be admitted to mathematical discourse. In this view, mathematics is an exercise of the human intuition, not a game played with meaningless symbols. Instead, it is about entities that we can create directly through mental activity. In addition, some adherents of these schools reject non-constructive proofs, such as a proof by contradiction.

1.4.2.9 Fictionalism

Fictionalism in mathematics was brought to fame in 1980 when Hartry Field published *Science Without Numbers*, which rejected and in fact reversed Quine's indispensability argument.[62] Where Quine suggested that mathematics was indispensable for our best scientific theories, and therefore should be accepted as a body of truths talking about independently existing entities, Field suggested that mathematics was dispensable, and therefore should be considered as a body of falsehoods not talking about anything real. He did this by giving a complete axiomatization of Newtonian mechanics that did not have reference numbers or functions at all. Having shown how to do science without using mathematics, he proceeded to rehabilitate mathematics as a kind of useful fiction. He showed that

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[61] Refer Paul Ernest, *Social Constructivism as a Philosophy of Mathematics*, State University of New York Press, Albany, NY, 1998, 52ff.

[62] *Ibid*., 626.

mathematical physics is a conservative extension of his non-mathematical physics that is, every physical fact provable in mathematical physics is already provable from his system, so that the mathematics is a reliable process whose physical applications are all true, even though its own statements are false. Statements are true according to the relevant fictions.

By this account, there are no metaphysical or epistemological problems special to mathematics. The only worries left are the general worries about non-mathematical physics and about fiction in general. Field's approach has been very influential, but is widely rejected. This is in part because of the requirement of strong fragments of second-order logic to carry out his reduction, and because the statement of conservativity seems to require quantification over abstract models or deductions. Another objection is that it is not clear how one could have certain results in science, such as quantum theory or the periodic table, without mathematics. If what distinguishes one element from another is precisely the number of electrons, neutrons and protons, how does one distinguish between elements without a concept of number?

1.4.2.10 Embodied Mind Theories

Embodied mind theories hold that mathematical thought is a natural outgrowth of the human cognitive apparatus which finds itself in our physical universe. For example, the abstract concept of number springs from the experience of counting discrete objects. It is held that mathematics is not universal and does not exist in any real sense, other than in human brains. Humans construct, but do not discover, mathematics.[63] With this view, the physical universe can be seen as the ultimate

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[63] Stewart Shapiro, *Thinking About Mathematics: The Philosophy of Mathematics*, Oxford University

foundation of mathematics: it guided the evolution of the brain and later determined which questions this brain would find worthy of investigation. However, the human mind has no special claim on reality or approaches to it built out of mathematics. Embodied mind theorists explain the effectiveness of mathematics that mathematics was constructed by the brain in order to be effective in this universe. The most famous advocates of this perspective are George Lakoff and Rafael E. Núñez and Keith Devlin.[64]

1.5 Conclusion

In this chapter, we have made a brief survey of the history of mathematics in view of placing Gödel's Theorems in the historical trajectory of mathematics. We have also articulated contemporary developments in the philosophy of mathematics with specific reference to the developments in twentieth century as a platform to delineate the relationship between mathematics and logic in general and also to expose the philosophical implications of *Gödel's Incompleteness Theorem* in particular. We have analyzed the philosophical schools of Mathematical Realism, Platonism, Logicism, Empiricism, the New Empiricism, Formalism, Intuitionalism, Fictionalism, Constructivism and Embodied Mind Theories to get a clear picture of the philosophical developments in mathematics as well as to explore the philosophical implications of mathematics.

What about the logicality of the statement of incompleteness theorem? Does it go beyond logical assumptions and presuppositions? Does logic itself experience a

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Press, Oxford, UK, 2000, 112. For details refer Alexandre George (ed.), *Mathematics and Mind*, Oxford University Press, Oxford, UK. 1994.

[64] Refer "Where Mathematics Comes from?," in http://en.wikipedia.org/wiki/Where_Mathematics_Comes_From, accessed on 08-01-2011.

‘limit’ when it is applied to Godel’s theorems? To what extent the Godelian incompleteness can be justified using Formalism which says true statements are assigned to the axioms and the rules of inference are truth-preserving? Is Godelian incompleteness real or fictitious? Mathematical realism and Fictionalism question the ‘reality’ of incompleteness? Similarly Intuitionism and constructivism raise questions regarding the truthness of a priori forms in mathematics. The following chapters address these concerns within the purview of fostering science-religion dialogue.