

The uses of mathematics in ancient Iraq, 6000–600 BC

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Introduction

Every culture has mathematics, but some have more than others. The cuneiform cultures of the pre-Islamic Middle East left a particularly rich mathematical heritage, some of which profoundly influenced late Classical and medieval Arabic traditions, but which was for the most part lost in antiquity and has begun to be recovered only in the last century or so.

Since city-living began to be the Middle Eastern way of life during the fourth millennium BC mud-brick, being cheap and plentiful, has been the principle urban building material for much of its history. But it is not particularly durable or weather resistant if it has not been baked, so that buildings have to be repaired or renewed every few generations. Inevitably, then, long-inhabited settlements gradually rise above their surroundings on a bed of rubbish and rubble as the centuries and millennia progress, forming mounds or ‘tells’. These tells are the raw material of archaeology, comprising successive layers of ever-older street plans, houses, domestic objects, waste pits—and written artefacts.

The centre of numerate, literate, urban culture in the pre-Islamic Middle East was southern Iraq, often called now by the Greek-derived term Mesopotamia, “between the rivers”, of the Tigris and Euphrates. The terms Sumer and Babylonia are also used, to refer to the area south of modern-day Baghdad in the third millennium BC and the second and first millennia BC respectively. Sumer was the cultural area of the speakers of a language called Sumerian, related to no other language known, which was first written down in the late fourth millennium and used, for an ever smaller number of functions, until the last centuries BC. It was gradually replaced by Babylonian, the southern dialect of the Semitic language Akkadian, named after the city of Babylon which was the region’s capital from the mid-eighteenth century BC for most of the following two millennia. In the north of Iraq, home of the Assyrian dialect of Akkadian, the city of Ashur was the cultural, religious and political centre from the late third millennium onwards, until a succession of more northerly capitals, most famously Nineveh, replaced it in the early first millennium BC.

Sumerian and Akkadian, although linguistically unrelated, shared a syllabic script we now call cuneiform, “wedge-shaped”, made by impressing a reed stylus on clay tablets which could range in size from a postage stamp to a laptop computer but were most often designed to be held comfortably in the hand. Clay is essentially an inorganic material and is not subject to the same decay processes as papyrus- or leather-based writing materials. When lost, abandoned, or thrown away in and around the mud-brick buildings of Mesopotamian cities tablets were for the most part preserved intact for millennia. Documents in languages such as Aramaic, on the other hand—written alphabetically with ink on perishable media, from the late second millennium onwards—survive rarely and badly.

We therefore rely primarily on cuneiform sources for our understanding of ancient Middle Eastern literate culture—and there is certainly no shortage of them. Conservative estimates put the number of clay tablets accessible to scholars in museum and university collections at around half a million world-wide. Many times that number are still in the ground. While all clay tablets are archaeological artefacts like pots or bones—for no texts have been passed on in continuous transmission—only a small fraction of those above ground have known archaeological contexts. In the early days of exploration and excavation, in the latter half of the nineteenth century AD, the aim of explorers and scholars was to fill Western collections with artefacts; the skills and methodology involved in the recovery of ancient environments and lifestyles had not yet been developed. Many thousands of tablets were dug up before the days of controlled, stratigraphic archaeology. But even during the twentieth century, excavators were slow to record detailed findspots for written artefacts, assuming that somehow they could ‘speak for themselves’ and did not need to be contextualised archaeologically. At the same time, particularly before the formation of the Iraqi Antiquities Authority in 1923, many sites were looted for their tablets, which were then sold on to private collectors and public institutions in Europe and America. As archaeological techniques were refined and excavation become both slower and more meticulous, and as specialist equipment became more expensive while budgets got smaller, digs became better managed, more limited in scope and focus. Since the Gulf War in 1990 the illegal trade in looted tablets and other antiquities has massively increased once more (Gibson, 1997). Inevitably, then, the number of tablets found in well-documented archaeological contexts is relatively small—but still can be counted in the tens of thousands.

A large percentage of those tablets, with context and without, deal with numbers. The vast majority are administrative documents, drawn up by bureaucrats working for large temples, palaces, or private enterprises which needed extensive and sophisticated literate and numerate management techniques to control their enormous resources. While these have been the bread and butter of social and economic historians, they have rarely been treated as sources for the history of mathematics. This is because we also have several thousand tablets which can strictly be called mathematical; that is, they deal with number or other mathematical matters for their own sake. While many are simple multiplication tables, others show a conceptual abstraction which is far removed from the practical arithmetical needs of everyday life. These mathematical tablets are known from the earliest phases of cuneiform culture to the latest, but date predominantly from the Old Babylonian period of the early second millennium BC.

Since its discovery in the early twentieth century AD, this mathematics has been treated implicitly as part of the ‘Western’ tradition; even now one finds ‘Mesopotamian’ mathematics categorised as ‘Early Western mathematics’, while Iraqi mathematics in Arabic, some of which is directly related to its compatriot precursors, appears under ‘Other traditions’ (e.g. Cooke, 1997). There is, however, no evidence of Mesopotamian influence on Classical mathematics until 150 years post-Euclid—despite a

century of determined attempts to show otherwise. To some extent this slant has been part of the mainstream European colonisation of the ancient civilisations. As Bahrani (1999: 163) puts it:

In the simplest terms, if the earliest ‘signs of civilisation’ were unearthed in an Ottoman province inhabited primarily by Arabs and Kurds, how was this to be reconciled with the European notion of the progress of civilisation as one organic whole? Civilisation had to have been passed from ancient Mesopotamia and Egypt to Greece.

But at the same time as ‘this unruly ancient time was [being] brought within the linear development of civilisation’ (Bahrani, 1999: 163) by dissociating it from the modern Middle East and grafting it instead to early European history, mathematics was itself being divorced from the history of the region for the simple reason that ‘few Assyriologists like numbers’ (Englund, 1998: 111). History of ancient Middle Eastern mathematics has, by and large, and certainly in the last fifty years, been left to mathematicians and historians of mathematics who have little feel for the culture which produced the mathematics or the archaeology which recovered the artefacts, and no technical training in the languages and scripts in which the mathematics was written (cf. Høyrup, 1996). Not surprisingly, the history of Mesopotamian mathematics has predominantly been a history of techniques and ‘facts’ about ‘what “everyone knew” in Babylon’ (Cooke, 1997: 47)—for the most part ‘translated’ beyond all recognition into modern symbolic algebra.

But the field is at last growing up, and since the late 1980s more serious efforts have been made to understand the language, conceptualisation, and concepts behind the mathematics of ancient Iraq (e.g. Høyrup, 1990a; 1994; Robson, 1999). This article is an attempt to pull the focus back further and to make a first approximation to a description of Mesopotamian numerate, quantitative, or patterned approaches to the past, present, and future; to the built environment and the agricultural landscape; and to the natural and supernatural worlds.

Decorative arts

Mathematics contains a strong element of the visual, making decorative designs and motifs one of the most pervasive and demotic sources for the identification of mathematical concepts within non-literate and extra-literate cultures (cf. Washburn and Crowe, 1988). Plotting the changing fashions in pottery design over the millennia is a key tool for the study of ancient Iraq—and indeed most archaeologically recovered societies—because fired clay is one of the most ubiquitous, malleable, replaceable, breakable and yet indestructible resources known to humankind. Changes in pottery style have been used to trace developments in society, technology, and artistic sophistication, but are also crucial witnesses to the place of geometrical concepts such as symmetry, rotation, tessellation, and reflection in the dominant aesthetics at all levels of a society.

Pottery-firing technology was in widespread use in northern Iraq from around 6000 BC. The earliest phases (so-called Samarra ware, cf. Figure 1) already exhibit strong geometric stylisation, as Leslie (1952: 60) notes:

A potter begins to decorate a pot with some notion of how the pot should look and with some ability [to] carry through this notion. Stated in formalistic terms, this notion of the Samarran potter is as follows:

1. The predominant use of unpaneled elements within enclosed bands.
2. The alternations of direction of movement and/or symmetrical motion of contiguous bands.
3. The emphasis of fourfold rotation or of fourfold radial symmetry of finite designs.
4. The dominant use of lineal rather than areal design elements.
5. The uniformity and precision of draftsmanship.
6. The tendency to balance equally painted and ground areas.

Stated in psychological terms, as an aesthetic ideal, the style is neat, busy, and abstract.



Figure 1: Samarra period bowl

c. 6000 BC, northern Iraq; painted clay; diameter 27.7 cm (after Caubet and Pouyssegur, 1998: 34)

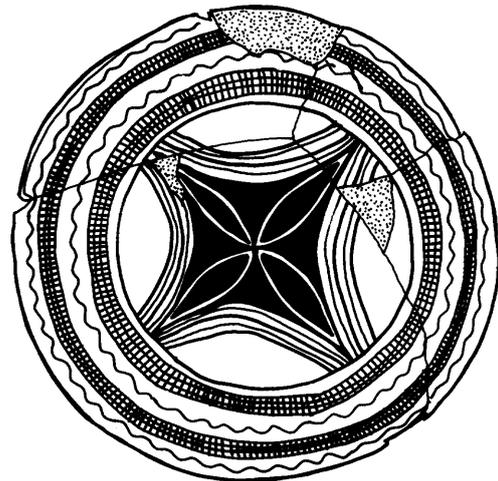


Figure 2: Halaf period bowl

c. 5000 BC, northern Iraq; painted clay; diameter 14 cm (after Reade, 1991: fig. 15)

These principles are exemplified by the well-preserved bowl shown in Figure 1. Around a central clockwise swastika four stylised herons catch fish in their mouths while eight fish circle round them, also in a clockwise direction. An outer band of stepped lines moves outwards anticlockwise, countering the swirling effect of the animalian figures.

We see similar geometrical concerns in the Halaf ware of the late sixth millennium BC. Figure 2 shows intersecting half- and quarter-arcs of circles forming symmetrical petal-like figures within hatched bands and wavy lines around the rim of the vessel. The overall impression is more static than the earlier Samarra ware. Kilmer (1990: 87) has drawn attention to the similarities with Old Babylonian geometrical figures called ‘cargo-boats’ composed of intersecting quarter-arcs (cf. Figure

10). Five- and six-fold radial symmetry are also occasionally attested in Samarra and Halaf ware (cf. Goff, 1963: figs. 34–35, fig. 69). Incidentally, potters at this early period are thought to have been almost exclusively women, working at home without potters' wheels, to supply their immediate household's domestic needs.

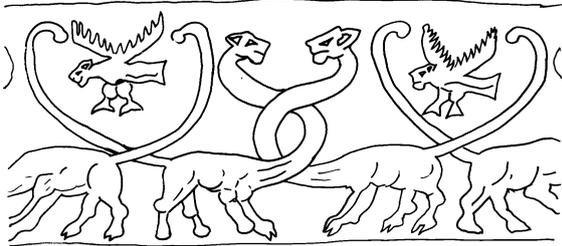


Figure 3: modern impression of cylinder seal
c. 3200 BC, unprovenanced; green stone;
4.6 × 4 cm (after Collon, 1987: no. 885)

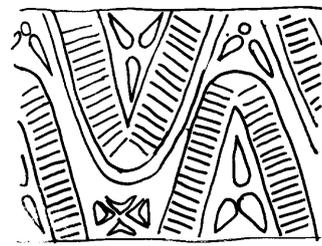


Figure 4: modern impression of cylinder seal
c. 3000 BC, south-western Iran; limestone;
4.0 × 1.3 cm (after Collon, 1987: no. 42)

Another key artefact type in the archaeology of ancient Iraq is the cylinder seal. These small stone objects, usually 3–5 cm in height and 1–3 cm in diameter, bore on their cylindrical surfaces incised designs which served to identify their owner or institutional function when rolled out onto the surface of clay covering vessel necks, knots, and other sealings. The original incisions had to be mirror-images of the intended clay impressions. Cylinder seals made their first appearance during the fifth millennium BC and, like pottery designs, showed strong a geometrical aesthetic from the earliest days, as well as a topological understanding of cylindrical surfaces, bounded at the top and bottom edges but unbounded on the curved plane. Most cylinder seal designs exploit this horizontal continuity, resulting in a seamless, never-ending design when rolled out on the more-or-less Euclidean surface of the clay. Figure 3 shows a sophisticated design of mirror-image fantastic animals whose elongated necks and tails intertwine and overlap each other, breaking up the vertical boundaries between the figures. Figure 4, on the other hand, exhibits a purely abstract continuous design whose main components are hatched arches which alternately descend and ascend from the edges of the seal, reaching almost to the opposite edge in two-fold rotational near-symmetry.

These devices of symmetry and continuity could equally well be used for abstract or figurative images. In Figure 5 we see pairs of birds (ducks?) with their wings outstretched within panels borders by twisted ropes. The image exhibits both horizontal and vertical symmetry. In Figure 6, water buffalo standing back-to-back drink from flowing water jars held by kneeling curly-haired men facing centre who are naked except for their cummerbunds. Underneath, a river runs continuously through a stylised mountainous terrain. The horns of the buffalo support a central cuneiform inscription in Sumerian recording the ownership of the seal: 'divine Shar-kali-sharri, king of Akkad: Ibni-sharrum, the scribe [and seal-owner] is your servant'.

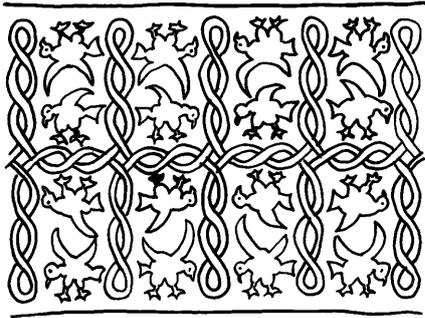


Figure 5: ancient impression of cylinder seal on clay, c. 3000 BC, Uruk, southern Iraq; height 6.4 cm (after Collon, 1987: no. 9)

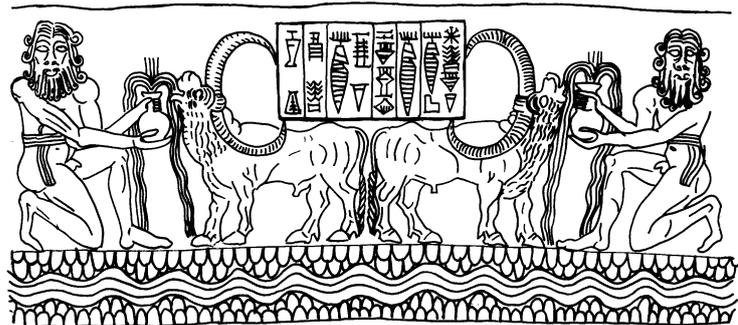


Figure 6: modern impression of cylinder seal c. 2200 BC, unprovenanced; jasper; 4.0 × 2.7 cm (after Collon, 1987: no. 529)

Symmetry could also be subverted or used to add meaning to an image. In Figure 7, a male human ruler, standing sideways, is mirrored to the right by a larger female deity facing front. The king pours offerings onto a stylised altar while the goddess offers the rod and ring, symbols of kingship, towards him. Trees and minor deities flank the main protagonists, but while the mountain gods are mirror-images one tree is crooked (and deciduous?) while the other is a straight pine. The symmetry is subverted by placing the gods to the left of each tree, but the overall feeling of balance—between human and divine, male and female, etc.—is maintained. The seal also carried a cuneiform inscription in Sumerian, enabling us to identify not only the owner of the seal but also the protagonists of the image. In this instance the text is not an integral part of the image but occupies the remaining third of the curved surface of the object, between pine tree and minor deity. It reads: ‘divine Amar-Suena, king of the four corners, beloved of the goddess Inana: Lugal-engardug—overseer of the temple of Inana, *nu-esh*-priest of the god Enlil, son of Enlil-amakh, overseer of the temple of Inana, *nu-esh*-priest of the god Enlil—is your servant.’



Figure 7: ancient impression of cylinder seal on clay; c. 2050 BC, Nippur, southern Iraq (after Gibson and Biggs, 1991: cover)

We have little evidence for mathematically inspired design on textiles and other bio-degradable objects, except for representations of them in less perishable media. The stone ‘carpets’ on the floors of major thresholds of the Neo-Assyrian palaces in northern Iraq are some of the most beautiful examples. Both of those shown here (Figure 8, Figure 9) are bordered with alternating open and closed symmetrical lotuses (giving a tasselled effect) and rows of rosettes. The central designs of Figure 8 are stylised flowers with four-fold symmetry, strongly reminiscent of the bowl in Figure 2, while the main ground of Figure 9 is composed of a series of interlocking circles forming petal-shaped figures from third-circle arcs.

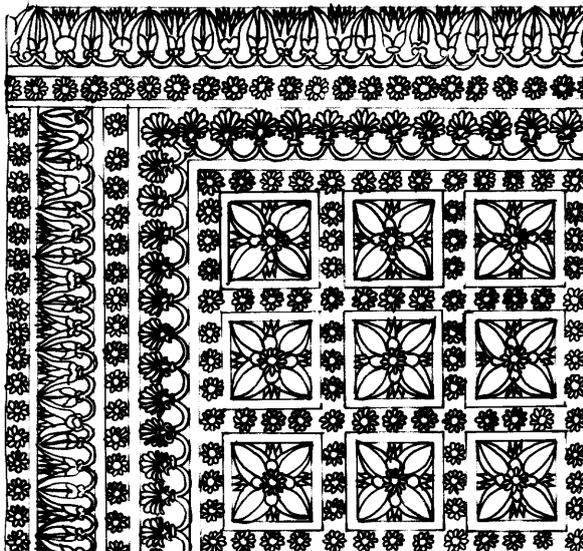


Figure 8: design on the door-sill of a palace throne-room, c. 685 BC; Nineveh, northern Iraq; Mosul marble; 45 × 45 cm (after Collon, 1995: fig. 114)

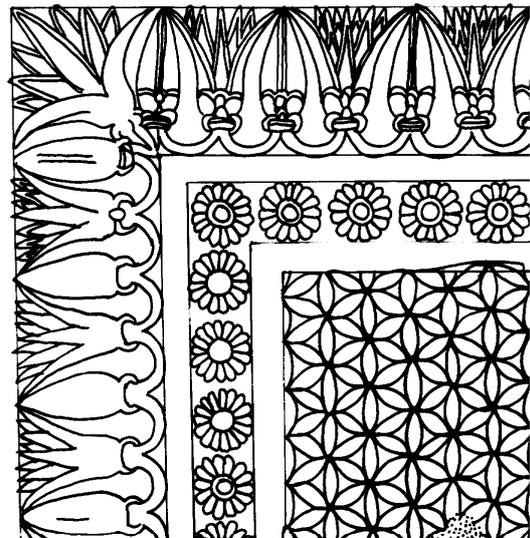
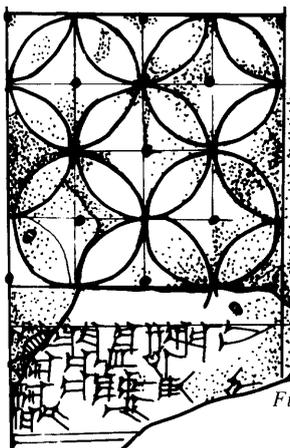


Figure 9: design on the door-sill of a palace throne-room, c. 645 BC; Nineveh, northern Iraq; Mosul marble; 65 × 65 cm (after Curtis and Reade, 1995: no. 45)

Two-, four- and eight-fold symmetry based on the square, with overlapping figures and shapes within shapes, is best attested for literate mathematics in the geometrical tablet BM 15285 (Figure 10), probably from the eighteenth century city of Larsa in southern Iraq (cf. Kilmer, 1990: 84–86; Robson, 1999: 34–56, 218–230). Geometrical figures based on the equilateral triangle and/or a third of a circle, however, are so far attested only in eighteenth century Eshnuna (east central Iraq) and seventeenth century Susa (south-west Iran) (Robson, 1999: 45–48).



“The side of the square is 60 rods (c. 360 m).
<Inside it are> 4 triangles, 16 barges, 5 concave
squares. What are their areas?”

Figure 10: The last preserved problem of the geometrical ‘text-book’ BM 15285
c. 1750 BC; Larsa, southern Iraq (after Robson, 1999: 217)

Bureaucracy and accountancy

Quantitative conceptual tools for managing and controlling property and wealth predate literacy by perhaps a thousand years (Englund, 1998: 42–55). During the course of the fourth millennium BC, temples were at the social and economic heart of developing urban centres such as Uruk and Susa. Through accumulation and close management of offerings (such as grain, cattle, land, precious stones, and metals) to deities they became wealthy and powerful institutions with interests in maintaining and increasing that wealth and power. Numerical methods of control afforded one such means (along with social methods such as propagation of religious ideologies). Small clay counters a few centimetres high, shaped into crude spheres, cones, and discs, appear to have represented fixed amounts of commodities within an accounting system; those commodities are no longer identifiable to us but the counters must have had context and meaning to the ancient administrators.

We can satisfactorily distinguish these counters from other small clay objects (such as sling-shots, loom weights, and beads) only when they are found archaeologically with other administrative artefacts such as cylinder seals, standard-sized mass-produced ration bowls, and the remains of the clay envelopes in which the counters were stored. More often than not they turn up not in the contexts in which they must have originally been used but in rubbish dumps within or in the vicinity of temple precincts. Small collections of counters are occasionally found within intact envelopes, which have sometimes been marked with cylinder seal impressions—marks of some institutional authority. Impressions of the tokens themselves may also be found on the surface of the envelopes—records of their contents (Nissen *et al.*, 1993: 11–13).

The oldest clay tablets known carry impressions of counters too, with or without cylinder sealings. We can trace a chronological and conceptual development over the fourth millennium: from pre-arithmetical tablets on which a single counter was impressed many times (perhaps as simple tallies); to those which show small numbers of counter-impressions of different shapes and sizes, and implicit

numerical relationships between them; to similar tablets which have been impressed not with a counter but with a stylus in the shape of a counter (Nissen *et al.*, 1993: 13–14). It would be tempting to posit an evolution from unmarked envelopes to those bearing impressions of counters, and from these to the arithmetical tablets—but the archaeological data does not (at the moment) support such a clear-cut chronological development.

Tablets with word signs as well as number signs probably date from the last third of the fourth millennium. The number signs continue to be made in imitation of the counters, while the words are for the most part pictograms scratched (not impressed) into the surface of the clay. Because each of the incised signs represents a whole word or idea rather than a particular sound, it is difficult (and perhaps inappropriate) to assign a language to these signs—but on the whole it is felt likely that the language is Sumerian. Nearly 4000 early tablets have been recovered from the foundations of the central temple of Uruk, where they had been re-used as building rubble. A Berlin-based team has studied them intensively, showing them to be elements of a sophisticated and complex system of managing the material wealth of the temple (Nissen *et al.*, 1993: 4–6).

The tablets use around a dozen different metrological systems and bases, dependent on the subject of the accounts. For instance, while most discrete objects, including nearly all dairy products, were counted in base 60, many others, including cheese, were counted in base 120. There were also separate methods for counting time, areas, and no less than six different capacity systems for various grains and liquids (Nissen *et al.*, 1993: 28–29). Because these number systems were contextual it was feasible to use visually identical signs in different numerical relationships: so that, for instance a small circular impression was worth ten small conical impressions in the discrete sexagesimal and bisexagesimal systems but sixteen in the areal system and just six in the barley capacity system. These relative values have been deduced from tablets on which several commodities are totalled together. Circular stylus-impressions on these early tablets are almost identical to the imprints of spherical counters on clay envelopes, while the conical impressions closely resemble the indentations made by conical counters. We can hypothesise, then, that the preliterate counters, like the early number signs, represented not absolute numbers but had the potential to embody a range of different numerical relationships dependent on what was being accounted for—and what those commodities were we are unlikely ever to know.

Even in those first tablets from the late fourth millennium, we find complex summations of different categories of goods, theoretical estimates of raw materials needed for food products, and the continuous tracking of grain harvests over several years (Nissen *et al.*, 1993: 30–46). The accounting year was assumed to be 360 days long, comprising 12 months of 30 days each, with an extra month inserted when needed to realign it with the seasons (Englund, 1988). This *ad hoc* intercalation was replaced by a standardised 19-year cycle of 235 lunar months only in the late first millennium BC (Rochberg, 1995: 1938). Over the course of the third millennium, writing was still used primarily for quantitative (and increasingly, legal) purposes, and its users were still mostly restricted to the

professional managers of institutional (temple and palace) wealth. In southern Iraq we have the names of individual accountants and administrators some 500 years earlier than the first royal inscription (cf. Postgate, 1992: 30, 66)! An important innovation of the later third millennium was the balanced account, in which theoretical outgoings were measured against actual expenditures—reaching a peak of complexity and ubiquity in the twenty-first century under the empire whose capital was the city of Ur (so-called Ur III). Under this highly managerial regime, accounts could be drawn up in silver (for merchants working for the state: Snell, 1982), grain (for millers), clay vessels (for potters), and even units of agricultural labour. Expected work- and production-rates were set at the upper limit of feasibility so that more often than not team foremen carried over a deficit of labour owed to the state from one accounting period to the next, year in year out (Englund, 1991).

Some time before the end of the third millennium the sexagesimal place value system (SPVS) was developed or invented. Metrologies, though rationalised periodically by administrative reform or royal decree (Powell, 1987–90), had continued to be context-dependent and to utilise many different number bases. This extract from the prologue to a 21st century law-code shows the increasing tendency to sexagesimalisation and cross-metrological relationships, where 1 *sila* \approx 1 litre and 60 *gin* = 1 *mana* \approx 0.5 kg:

I made the copper *bariga* and standardised it at 60 *sila*. I made the copper *ban* and standardised it at 10 *sila*. I made the regular royal copper *ban* and standardised it at 5 *sila*. I standardised the 1 *gin* metal weight against 1 *mana*. I made the bronze *sila* and standardised it at 1 *mina*. (after Roth, 1995: 16)

Nevertheless, it was not always a trivial matter to convert between systems—to calculate the area of field given its length and width, say, or to find the grain capacity of a given volume—and the SPVS seems to have been a calculational device created to overcome these difficulties. Measures were converted to sexagesimal multiples and fractions of a designated base unit and the calculation performed in base 60, the results of which were transformed back into the units of the appropriate metrology. Professional scribes were apparently not supposed to show their arithmetical workings, so that we get only occasional glimpses of the SPVS at work in the late third millennium (Powell, 1976).

By the early second millennium, with the further spread of writing into the personal sphere, we have a good deal of evidence about legal and financial uses of mathematics. The standard units of commercial exchange were barley (for items of low value) and silver (for more expensive goods). Law-codes set out ideal rates of exchange, wages, and professional fees (Roth, 1995: 23–142) but they were often much higher in practice than laid down in theory (Postgate, 1992: 195). Loans of barley were made at an average interest rate of 33 1/3%, silver at 20%. These rates were not annual but for the duration of the loan, however long it lasted—usually a matter of months. Willed property was measured out and divided in equal portions among the heirs (usually, but not exclusively, sons) with the eldest getting an extra share in recompense for performing *kispum* rituals for dead ancestors.

Women could own and manage property too, whether in their own right or on behalf of their families (Postgate, 1992: 88–108). Mathematical problems from the same period, however, do not reflect contemporary practice but rather use inheritance and loan scenarios to set up pseudo-realistic word problems on topics like arithmetic progressions and division by irregular numbers (Friberg, 1987–90: 569–570).

Quantity surveying and architecture

It is a very old chestnut indeed that mathematical innovation was driven in premodern societies by the need to measure fields and predict harvests. Nonetheless, there is a kernel of truth to it. We have already seen that the oldest known literate institution, the temple in Uruk, included quantitative land and crop management amongst its activities.

Once again, though, our best evidence is from the later third millennium. Southern Iraqi fields at this time were typically elongated strips, designed both to minimise the number of turns ploughing teams of oxen had to make and to maximise the number of fields abutting the irrigation channels. Field plans drawn up by the scribes of Ur show that irregularly shaped areas around the main arable lands were divided into approximately right triangles and irregular quadrilaterals, whose areas were calculated as the products of averaged opposite sides (Liverani, 1990). The diagrammatic elements of the plans are never to scale; rather, we might say that they are relational or topological, showing only the spatial relationships between the field elements and their basic geometrical shapes. All quantitative information was contained in the annotations to the plans: the measurements of the fields' sides, their calculated areas, and sometimes one or more cardinal points at the edges of the plan. The text could also contain descriptive information, such as the quality of the soil or the names of the fields. From such plans theoretical harvest yields could be calculated, and compared to those actually achieved.

Agricultural labour was also closely managed from at least the mid-third millennium on, with workers being allotted target work-rates for tasks such as renovating irrigation canals or weeding. Under the Ur III regime, such work-rates were used as units of account in a system of annual double-entry bookkeeping. In the region of Umma in southern Iraq teams of 20, including an overseer, were expected to contribute 7,200 working days a year, plus whatever was owed from the year before. Administrators kept records of the work they performed, following the agricultural cycle from the spring-time harvest (reaping, making sheaves, threshing); through field preparation (ploughing and harrowing in teams of three or four, at 1.8 ha a day, sowing at 0.7 ha a day); hoeing and weeding (360–1,080 m² a day); to repairing channels and banks (1.6–3 m³ a day) (Maekawa, 1990; Robson, 1999: 157–164). At the end of the accounting year, the work completed was compared with the work expected, and any deficit carried over to the following accounting period (Englund, 1991). Many of these theoretical labouring rates were still used to set word problems in Old Babylonian school mathematics, even though most had long fallen out of practical use (Robson 1999: 93–110).

Building labour was managed in a similar way. One of the earliest extant building plans, and in some ways the most informative, is not a working document at all but a sculptural depiction of one (Figure 11). It is the focal point of an inscribed statue, one of a series of around twenty, depicting Gudea, who ruled a small state in southern Iraq at the end of the 22nd century BC. The inscription tells us that the city god Ningirsu had revealed to Gudea in a dream the layout of a new temple for him, the enclosure walls of which are outlined on the plan. It goes on to describe the construction of the temple, and how the statue was to be set up in its principal courtyard facing the statue of the god himself (Tallon, 1992: 41). Along the outer edge of the plan, the remains of a ruler are just visible. The ancient temple itself has been excavated, but sadly the statue was not recovered in situ; it was found with seven others in the ruins of a palace on the same site, built in the second century BC—placed there some two thousand years after its manufacture!

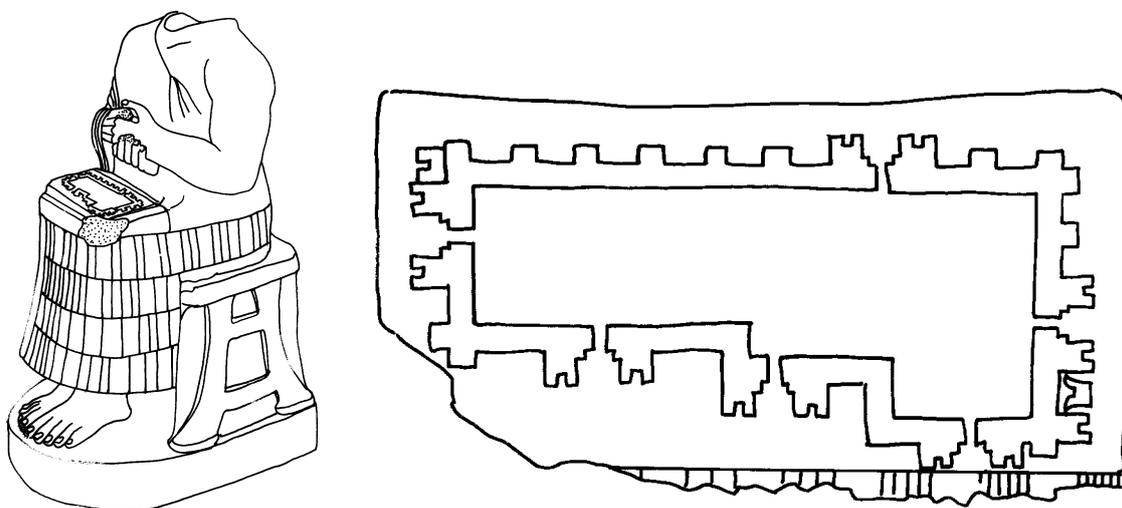


Figure 11: Statue of Gudea, city governor of Lagash, with a building plan and ruler on his lap
c. 2100 BC, southern Iraq; diorite; statue 93 × 46 × 62 cm, plan 26 × 16 cm
(after Tallon, 1992: 42–43, figs. 12, 12a)

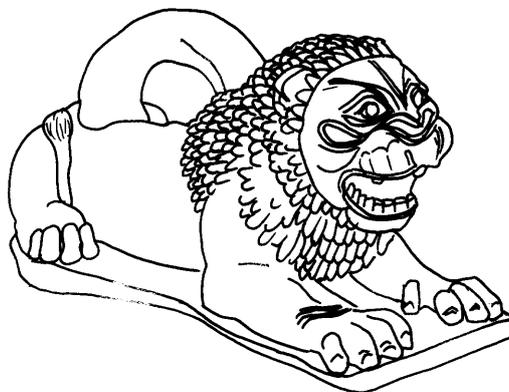
We also have less monumental house plans on clay tablets from the late third and early second millennia BC (Postgate, 1992: 91, 117; Robson, 1999: 148–152), which like the field plans give measurements and sometimes descriptions of the functions of the rooms. Contemporary administrative documents listing walls to be built and repairs to be made show that there were two standard sizes of brick. Baked bricks, which look more like paving stones to us, but were used for prestige buildings and for flooring, measured $\frac{2}{3}$ cubit (c. 33 cm) square; the cheaper sun-dried bricks, on the other hand, were $\frac{1}{2} \times \frac{1}{3}$ cubit (c. 25 × 17 cm). Both were 5 fingers (c. 8 cm) thick. Bricks, whatever their size, were counted in groups of 720. The number of 720s per unit volume, or brickage—2.7 for square baked bricks, 7.2 for sun-dried bricks—was a useful constant in calculating materials for building work. Sometimes mortar was factored in as $\frac{1}{6}$ of the volume of a wall, in which case constants of 2.25 and 6 were used instead (Robson, 1999: 145–148). Standardised brick sizes were elaborated into

a complex metrology in Old Babylonian school mathematics, with about a dozen types attested in many different sorts of word problems (Robson, 1999: 57–73). The theoretical brick sizes are very close to the measurements of ancient bricks recovered from Iraqi archaeological sites, where the square baked bricks are attested more-or-less continuously from the time of Gudea to the Persian period (sixth century BC) (Robson, 1999: 278–289).

Education

Not surprisingly, mathematics education is as old as mathematics itself. Even amongst the earliest cuneiform tablets from late fourth millennium Uruk we find exercises in bookkeeping and calculation which exhibit features we might expect from ‘school work’: they are anonymous, often incompetently executed, and result in conspicuously whole or round numbers (Englund, 1998: 106–110). Word problems, mathematical diagrams, and arithmetical tables are attested patchily from the mid-third millennium onwards (Friberg, 1987–90: 540–542), but our best and most abundant evidence for mathematics education comes from the Old Babylonian period. A few hundred cuneiform tablets between them contain over a thousand word problems, while the number of arithmetical tables surviving must run well into the thousands too. This is the subject matter of most modern surveys of ‘Babylonian’ mathematics (e.g. Neugebauer and Sachs, 1945; Friberg, 1987–90: 542–580; Høyrup, 1994; cf. Robson, 1996–) so my aim here is not particularly to summarise its contents but to explore how and why it was taught and learned. Most of the evidence we have comes from the eighteenth century cities of Nippur and Ur, between modern Baghdad and the Gulf (cf. Tinney, 1998).

Trainee scribes’ first encounter with mathematical concepts was in the course of copying and learning by heart a standard list of Sumerian words, organised thematically, and running to over 2000 entries. In the section on trees and wooden objects, for instance, the subsection on boats includes eight lines on boats of different capacities (1 *gur* \approx 300 litres): “60-*gur* boat; 50-*gur* boat; 40-*gur* boat; 30-*gur* boat; 20-*gur* boat; 15-*gur* boat; 10-*gur* boat; 5-*gur* boat”. Similarly, the section on stone objects included a list of about thirty weights from 1 *gun* to 5 *gin* (\approx 30 kg – 4 g) (cf. Figure 12).¹



¹ The Assyrian royal *mana* of the early first millennium was double that of the traditional *mana*, which also continued to be used (Powell, 1987–90: 516).

Figure 12: Bronze weight, inscribed in Akkadian “Palace of Shalmaneser, king of Assyria, 5 royal *mana*”. One of a set of eight, *c.* 725 BC. Nimrud, northern Iraq, 10.5 × 20 × 8 cm, 5.04 kg (after Curtis and Reade, 1995: no. 202)

They also learned weights and measures in a more structured fashion, writing out standard lists of capacity, weight, area, and length, in descending order of size, often with their SPVS equivalents (Friberg, 1987–90: 542–543). Equally, they copied a series comprising a division (‘reciprocal’) table followed by multiplication tables for forty sexagesimally regular numbers from 50 down to 2 (Friberg, 1987–90: 545–546). The procedure for all rote learning, whether of Sumerian words or arithmetical facts, was the same: first the teacher wrote out a model of 20–30 lines for the student to repeatedly copy on the same tablet; then the student progressed to writing out extracts on small tablets; and finally the whole series was written out on one, two, or three large tablets (Veldhuis in Tinney, 1998). Addition and subtraction facts, however, were never committed to memory in this way.

Scribal students also had the opportunity to practice their arithmetical skills, making calculations or drawing geometrical diagrams on small round or roughly square tablets (Robson, 1999: 245–277). In many instances we can link those calculations with particular problem-types which are set and given model solutions on other tablets, which we might think of as ‘textbooks’ (cf. Figure 10; e.g. Fowler and Robson, 1998: 368–370). Those model solutions are essentially instructions in Akkadian, using as a paradigm a convenient set of numerical data which will produce an arithmetically simple and pleasing answer. Around half of such problems use ‘real life’ scenarios such as the building, labouring, and inheritance contexts described above, but they should by no means be considered examples of narrowly ‘practical’ mathematics. Even when they use constants (brick sizes, labouring rates, etc.) known from administrative contexts, the problems themselves are clearly impractical: to find the length and width of a grain-pile given their sum, for instance, or to find the combined work-rate of three brick-makers all working at different rates (Robson, 1999: 75, 221). In short, these scenarios are little more than window-dressing for word problems involving mathematical techniques that working administrators and accountants were almost certainly never likely to need.

The remaining problems have traditionally been classed as ‘algebra’ and translated into arithmetised x - y symbolism. Jens Høyrup’s groundbreaking study of Akkadian ‘algebraic’ terminology has shown, however, that the scribes themselves conceptualised unknowns much more concretely as lines, areas, and volumes (Høyrup, 1990a). These imaginary geometrical figures could then be manipulated, as described in the model solutions, until the magnitude of the unknown was found using techniques such as completing the square (cf. Fowler and Robson, 1998). We see then, that in the Old Babylonian period number had not yet completely shed its contextualised origins of 1500 years before: not only did it have magnitude but it still had dimension or measure. Just as administrative scribes recorded data and results in mixed metrological systems and used the SPVS only for intermediate calculations, even in the ‘pure’ setting of school mathematics the inherently

metric (measured) properties of number were disregarded in favour of the SPVS solely for the duration of arithmetical operations.

Literacy, and literate numeracy, were professional skills in ancient Iraq, possessed with greater or lesser degrees of competence by a tiny percentage of the urban population. But if scribal education was no more than a vocational training why, then, did Old Babylonian school mathematics far exceed the needs of working scribes, most of whom would have spent their lives as accountants or letter writers in bureaucratic institutions? One could ask the same of other aspects of their education: the long lists of rare and complex cuneiform signs, for instance, or the Sumerian literary epics. Some tentative answers may be found in the very subject-matter that the students studied. A good many of the Sumerian proverbs (often found on the same tablets as arithmetical exercises: Robson, 1999: 246) and literary passages directly concerned or alluded to scribalism: the attributes of accomplished scribes were elaborated and extolled, while those of the incompetent were derided. High levels of literacy and numeracy were worthy of the most renowned of kings, endowed by Nisaba, goddess of scribal wisdom:

Nisaba, the woman radiant with joy, the true woman, the scribe, the lady who knows everything, guides your fingers on the clay: she makes them put beautiful wedges on the tablets and adorns them with a golden stylus. Nisaba generously bestowed upon you the measuring rod, the surveyor's gleaming line, the yardstick, and the tablets which confer wisdom. (Praise poem of king Lipit-Eshtar (Lipit-Eshtar B), lines 18–24. Black *et al.*, 1998–: no. 2.5.5.2)

In short, the true scribe was not merely competent; he possessed divinely bestowed skills and wisdom which far exceeded the humdrum needs of his daily life, yet were still somehow related to it (cf. Høyrup, 1990*b*: 67).

Divination

The prediction of the future through the observation of ominous phenomena does not seem at first sight to be an activity rich in mathematical thinking. Extispicy, or divination by inspection of the livers of dead sheep, arose from the need to feed the gods, who in some sense inhabited beautiful statues of themselves housed in the temples of Mesopotamian cities. The gods fed, so the idea went, by smelling the offerings of food, drink, and incense made to them. These same gods, it was thought, decided the future of the world and recorded it all in cuneiform on the Tablet of Destinies. But they revealed their intentions in subtle ways, in particular—if the correct rituals were performed—in the entrails and especially in the livers of the sheep and goats sacrificed to them. If the expert diviners determined that the future the gods had in store was unfavourable, they could take measures to avert it. This was also a task for experts, and involved further prayers and rituals and sacrifices to the gods in order to persuade them to change their minds. A further liver divination would determine whether the procedure had been effective or not. The first clear evidence for liver divination comes from the end of

the third millennium, when some years were named after high priests being chosen in this fashion. As well as these revealed omens, various happenings in the natural world and skies—so-called observed omens—could also be considered portentous.

Our richest sources of evidence are the compendia of omens, which are first attested in the early second millennium BC and reach their most elaborate and comprehensive form in eighth and seventh century Assyria (northern Iraq). By that time the omens had been collected into a series of around 100 tablets, divided into ten chapters. Its Akkadian name is *bârûtu* ‘seeing’. The first nine chapters are organised according to the ominous organs of the body, and work systematically through various features and defects of each. The omens operate by two sorts of general principle. First there are binary oppositions, such as right-left, up-down, large-small, and light-dark. Right was associated with a propitious omen, left with an unpropitious one. So while good health on the right side of the animals innards was auspicious, on the left it was inauspicious. Conversely, abnormalities were hoped for on the left, but not on the right, as in this extract from a divination prayer:

Let the judges, the great gods, who sit on golden thrones, who eat at a table of lapis-lazuli, sit before you. Let them judge the case in justice and righteousness. Judge today the case of so-and-so, son of so-and-so. On the right of this lamb place a true verdict, and on the left of the lamb place a true verdict. [...] Let the back of the lung be sound to the right; let it be stunted to the left. Let the ‘receptacle’ of the lung be sound on the right; let it be split on the left. (Starr, 1983: 37–38, lines 18–19, 30–31)

Secondly, various sorts of analogies were also used to make predictions. Punning played a prominent part, as did the association of specific features or dispositions of the internal organs with phenomena in the real world: mildew presaged ruin, berry-like excrescences portended warts, etc. Certain fortuitous markings had exact meanings too: the Weapon foretold war and death, the Foot suggested movement, and the Hole forecast death and disaster.

Particular features of the entrails could also be identified with individuals or institutions in the real world: on the liver the Palace Gate referred to the palace of course, and the Path, meant the army on campaign. These associations were explicitly stated in the last of the ten chapters of the *bârûtu* omen series, which was called *multâbiltu*, or ‘analysis’. Various attributes of the organs—length, thickness, massiveness, movement—were each linked with a general prediction—health, fame, power, happiness—and illustrated with an omen extracted from the first nine chapters (Starr, 1983: 6).

We see similar principles at work in collections of omens observed from real world phenomena, for instance the series concerning ominous births called *shumma izbu*, ‘if a birth-anomaly’. This extract concerns the birth of kids which are a different colour to their mother goats:

If a black goat gives birth to a yellow kid: that fold will be scattered; it will become waste; there will be anger from the gods.

- If a yellow goat gives birth to a black kid: that fold will be scattered; ...
 If a white goat gives birth to a black kid: destruction of the herd.
 If a black goat gives birth to a white kid: the fold of the man will scatter.
 If a red goat gives birth to a black kid: destruction of the herd.
 If a black goat gives birth to a red kid: destruction of the herd.
 (Leichty and von Soden, 1970: 175)

It is the birth of a black kid to a non-black goat that is unfavourable, as can be seen when the combinations of kids and goats of different colours are tabulated. Presumably other sorts of births are not ominous, for good or bad.

<i>Goats/Kids</i>	black	yellow	white	red
black		•	•	•
yellow	•			
white	•			
red	•			

But omens do not restrict themselves to a systematic, exhaustive enumeration of possible outcomes. Because, we remember, omens are signs from the gods, in theory almost anything is possible. So we find, for instance, goats giving birth to lions, wolves, dogs, and pigs. Similarly, the tablets of *shumma izbu* about other animals list equally impossible—better, improbable—birth events. Nevertheless, despite the lack of causal relationship between observation and prediction, we can detect some basic combinatoric and group theoretical concepts behind the development of divination.²

Epilogue and conclusions

The chronological cutoff point for this survey roughly coincides with the end of the Assyrian empire in 612 BCE. A few decades after that point Iraq became little more than a province of the Persian empire (albeit a wealthy and important one), followed by periods of Alexandrian, Parthian, and Sassanian rule. It did not regain its political independence and cultural dominance until the rise of Islam in the sixth century CE. Nevertheless neither cuneiform nor mathematics died a sudden death (Friberg, 1993; Geller, 1997). On the contrary, both gained a new lease of life with the increasing cultic importance of celestial divination, and the concomitant need to predict accurately the movements of the heavenly bodies. The detailed observations of heavenly bodies recorded in Babylon over the course of the first millennium BCE, the arithmetical schemes used to model their movements, and not least the SPVS in which those observations were recorded, were all crucial building blocks for

Hipparchus and Ptolemy to lay the foundations of modern astronomy in turn-of-the-millennium Egypt (Toomer, 1988; Jones, 1993). Indeed, the SPVS (though not in cuneiform) remained the only universal and viable vehicle for astronomical and trigonometrical calculations until the coming of the Indian-Arabic decimal system that we use today. Old Babylonian-style cut-and-paste ‘algebra’ also heavily influenced the mathematical work of Diophantus and Hero in late Classical Egypt as well as early Islamic algebraists, although the direct links in this case are much more difficult to detect (Høyrup, 1994).

In summary, then, numerical and mathematical concepts were an integral part of the scribal world view, running throughout ancient Iraqi literate culture (and non-literate culture too). The ancient Middle East witnessed not merely the ‘infancy’ of Western mathematical culture—though, as we have seen, it played a major role in the birth of astronomy—but also in a very real way housed the intellectual precursors of the marvellous flowering of Arabic mathematics in the early Middle Ages. Without this of course, much of the Classical mathematics we hold so dear would not have survived at all, and trigonometry, algebra, and algorithms—all of which have strong roots in the mathematics discussed here—might have looked very different, if they existed at all.

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² The famous Babylonian mathematical astronomy of the late first millennium BC also had its roots in divination through observation of celestial phenomena (Koch-Westenholz, 1995; Rochberg-Halton, 1995; Veldhuis, 1999), but it is beyond the chronological scope of the article, and the competence of the author, to trace that development here.

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