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Was Babylonian Mathematics Created by `Babylonian Mathematicians´?

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Already Moritz Cantor would speak [1907: 19–51] without hesitation about "Babylonian mathematics"; as it turned out around 1930 that this mathematics went far beyond the use of a place value system, tabulation of squares, products and reciprocals, and the determination of simple areas, it became habitual to speak of its practitioners as "Babylonian mathematicians" (which Cantor had not done except in a quotation).

If mathematicians are understood as people who excel in making more complex numerical computations than the rest of the human race (an idea which contemporary mathematicians claim to encounter regularly at dinner parties and on similar occasions), then the notion of "Babylonian mathematicians" is certainly no scandal.

However, these same mathematicians *are* scandalized by the ignorance of the dinner neighbour. They may not insist that the essence of their trade is to make demonstrations – also because they know that creative mathematicians get their good ideas first and make their more or less appropriate proofs afterwards, often leaving perfection to later workers. They may also admit that applied mathematics – mathematical statistics, mathematical hydrodynamics, etc. – should count as mathematics. But somehow they will insist that the mathematician creates insights in the formal properties of mathematical objects, correlates the properties of different mathematical objects or classes of objects, finds overarching theoretical structures, or something similar.[1]

In this sense, Euclid and Archimedes were certainly mathematicians, and so were those Pythagoreans (called, precisely, μαθηματικοί) who in the fifth century BCE explored the properties of "the odd and the even" and of triangular and square numbers. But what about the authors of the Babylonian mathematical texts?

Asking for the direct aim of the texts we find little or nothing that suggest a "mathematician's intention". We may leave aside both mathematical tables and tablets for rough numerical work – the former are aids for numerical computation, the latter train it. The third category is constituted by problem texts, containing either a sequence of problem statements alone (at times also with indication of the solution) or one or more problem statements followed by prescriptions. From the third millennium we have a few student texts indicating a problem and the corresponding solution,[2] the second- and third-millennium specimens are teachers' copies.

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[1] This characterization, we may observe, also serves to distinguish the mathematician from the numerologist and his kin. Numerology and related schemes correlate the properties of single mathematical objects with those of non-mathematical objects (the perfection of the number 6 with the duration of the Creation, the triangle with Trinity, etc. This kind of correlation between single objects should be distinguished from that mapping of mathematical *structures* on real-world structures which is the basis of any applied mathematics.

[2] From the proto-literate period and Ur III we have a number of administrative model documents and no other mathematical school texts; evidence from the Old Babylonian vocabulary suggests that at least Ur III produced no other mathematical school texts – cf. [Høyrup 2001].

Some of the problems train the solution of problems of direct practical relevance for the future scribe; others, though apparently dealing with similar matters (dimensions of fields, constructions and excavations, prices, brick production and workmen's wages, etc.) turn out at closer inspection to treat of situations that could never arise in non-school practice – to find the side of a square field when the sum of the sides and the area is known, or to find the rates (inverse prices) at which a given quantity of oil is bought and sold if the total profit and the difference between the rates is given.

Such texts are particularly conspicuous in the Old Babylonian record, where we also find the most sophisticated expressions of the "supra-utilitarian" interest. The third millennium offers only rather unapparent beginnings of this trend, and the first-millennium examples are few, as are first-millennium mathematical texts in general. I shall therefore restrict the discussion to the Old Babylonian period.

Is it then justified too see the supra-utilitarian problems of the Old Babylonian period as expressions of "mathematician's intentions"? Firstly, we may observe that supra-utilitarian no less than utilitarian problems aim at *finding the right number*. In one case as in the other, solutions presuppose mathematical insights, and part of the aim of having students solve numerous problems of more or less identical structure may well have been to impart such insights in an informal way; but the utterly few examples we possess of texts involving didactical explanation of the meaning of operations and intermediate results[3] seem to show that such insights were not made explicit; the same conclusion follows from the kind of proofs that are sometimes given – namely numerical control of the agreement of the result with the statement. In some early Old Babylonian texts we also find rules formulated in abstract terms or reference to such rules[4], but these are wholly devoid of explanation. At this level, no argument impels us to speak of the authors of the Old Babylonian mathematical texts as "mathematicians" (nor, certainly, as numerologists). We should rather see them as "teachers of computation", at times of unapplicable computation; the impartation of insight remained ancillary to this aim, in agreement with this passage from Christian Wolff's *Mathematisches Lexicon* [1716: 867, trans. JH]

It is true that performing mathematics [*ausübende Mathematick*] can be learned without

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[3] Among published texts, the Susa texts TMS VII, IX and XVI contains very specific didactical expositions, while YBC 8633 is less direct. An unpublished texts from Eshnunna (IM 43993) is similar is this respect to the Susa texts. See [Høyrup 2002: 85–95, 181–188, 254–257].

[4] The proof of Db_2-146 quotes the "Pythagorean rule" for determining the diagonal of a rectangle; AO 6770 #1 and IM 52301 #3 are very opaque formulations of general rules – so opaque, indeed, that it becomes understandable why the use of such rules was given up in the later Old Babylonian period (the Late Babylonian text W 23291 couples general rules with illustrative paradigmatic examples, which makes the rules intelligible).

The chronological ordering of the Old Babylonian mathematical corpus is discussed in [Høyrup 2000], and (with inclusion of further texts from Ur and Nippur) in [Høyrup 2002: 317–361].

reasoning mathematics; but then one remains blind in all affairs, achieves nothing with suitable precision and in the best way, at times it may occur that one does not find one's way at all. Not to mention that it is easy to forget what one has learned, and that that which one has forgotten is not so easily retrieved, because everything depends only on memory. Therefore all master builders, engineers, calculators, artists and artisans who make use of ruler and compass should have learned sufficient reasons for their doings from theory

– only with the difference that "theory" proper apparently did not exist in the Old Babylonian epoch.

At a different level, however, it may perhaps be legitimate to speak of these teachers (or some of them) as mathematicians in a sense which corresponds to later usage. In order to see that we shall first look at some texts, and next ask for the motives that called for the teaching of unapplicable computation.

One text of interest is AO 8862 #1:[5]

1. Length, width. Length and width I have made hold:
uš sağ uš *ù* sağ *uš-ta-ki-il*$_5$*-ma*

2. A surface have I built.
a.šà^{lam} *ab-ni-i*

3. I turned around (it). As much as length over width
as-sà-ḫi-ir ma-la uš *e-li* sağ

4. went beyond,
i-te-ru-ú

5. to inside the surface I have appended:
a-na li-ib-bi a.šà^{lim} *u-ṣi-ib-ma*

6. 3`3. I turned back. Length and width
3.3 *a-tu-úr* uš *ù* sağ

7. I have accumulated: 27. Length, width, and surface w[h]at?

27	3`3	the things accumulated
15	the length	3` the surface
12	the width	

ğar.ğar-*ma* 27 uš sağ *ù* a.šà *mi*-[*nu*$^?$]-*um*

27	3.3	*ki-im-ra-tu-ú*
15	uš	3 a.šà
12	sağ	

8. You, by your proceeding,
at-ta i-na e-pe-ši-ka

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[5] Ed. [MKT I, 108*f*]. The translation is mine (as are all following translations of Babylonian material), and borrowed from [Høyrup 2002: 164*f*]. This volume also explains the principles governing my "conformal translation". I follow Thureau-Dangin's transcription of sexagesimal numbers, in which `, ``, ... indicate increasing and ´, ˝, ... decreasing sexagesimal order of magnitude; 3`3 is thus equal to 3·60+3 = 183, 14°30´ to $14+{}^{30}/_{60}$.

9. 27, the things accumulated, length and width,
27 *ki-im-ra-at* uš *ù* sag̃

10. to inside [3`3] append:
a-na li-bi [3.3] *ṣi-ib-ma*

11. 3`30. 2 to 27 append:
3.30 2 *a-na* 27 *ṣi-ib-ma*

12. 29. Its moiety, that of 29, you break:
29 *ba-a-šu ša* 29 *te-ḫe-ep-pe-e-ma*

13. 14°30´ steps of 14°30´, 3`30°15´.
14.30 a.rá 14.30 3.30.15

14. From inside 3`30°15´
i-na li-bi 3.30.15

15. 3`30 you tear out:
3.30 *ta-na-sà-aḫ-ma*

16. 15´ the remainder. By 15´, 30´ is equal[side.]
15 *ša-pi-il*$_5$*-tum* 15.e 30 íb.[si$_8$]

17. 30´ to one 14°30´
30 *a-na* 14.30 *iš-te-en*

18. append: 15 the length.
ṣi-ib-ma 15 uš

19. 30´ [fr]om the second 14°30´
30 [*i*]*-na* 14.30 *ša-ni-i*

20. you cut off: 14 the width.
ta-ḫa-ra-aṣ-ma 14 sag̃

21. 2 which to 27 you have appended,
2 *ša a-na* 27 *tu-us*$_4$*-bu*

22. from 14, the width, you tear out:
i-na 14 sag̃ *ta-na-sà-aḫ-ma*

23. 12 the true width.
12 sag̃ gi.na

24. 15, the length, and 12, the width, I have made hold:
15 uš 12 sag̃ *uš-ta-ki-il*$_5$*-ma*

25. 15 steps of 12, 3` the surface.
15 a.rá 12 3 a.šà

26. 15, the length, over 12, the width,
15 uš *e-li* 12 sag̃

27. what goes beyond?
mi-na wa-ta-ar

28. 3 it goes beyond. 3 to inside 3` the surface append,
3 *i-te-er* 3 *a-na li-bi* 3 a.šà *ṣi-ib*

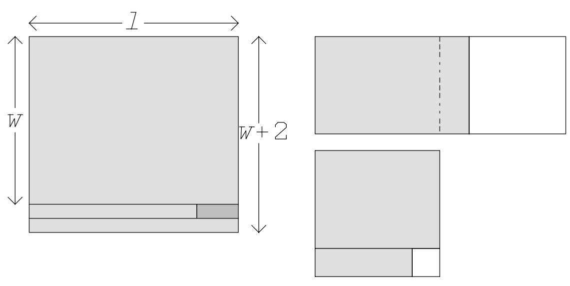

Figure 1. The situation and procedure of AO 8862 #1.

29. 3`3 the surface.
3.3 a.šà

The problem, as shown to the left in Figure 1, deals with the simplest figure that is determined from a single length (uš) and a single width (sag̃) – that is, according to Babylonian habits, a rectangle. These dimensions ℓ and w are made "hold" each other (*šutakūlum*), and thus a rectangular "surface" or field (a.šà*lam*~*eqlam*) is "built" (*banûm*) or constructed. To this rectangle the excess of the length over the width is "appended" (*waṣābum*) or joined (heavily shaded in the diagram). This joining presupposes that ℓ and w are understood as "broad lines", lines provided with a virtual breadth equal to the length unit (the nindan). The resulting area is told to be 3`3. We are also told the "accumulation" or arithmetical sum of the two sides (addition by the verb *kamārum*). Joining these (still "broad") to the configuration gives us a new rectangle with width $W = w+2$ and length $\ell$ – whence $\ell+W = 27+2 = 29$, while the area is 3`3+27 = 3`30.

Thereby we are brought back to a standard problem, that of finding the sides of a rectangle from the area and the sum of the two sides. The procedure is shown to the right in Figure 1: the sum of L and w is "broken" (*ḫepûm*), that is, bisected and rearranged so as to contain a square; each piece is evidently the average $\frac{\ell+W}{2}$ = 14°30´; the area of this square is found as 14°30´·14°30´, the multiplication involved being the one used in the tables of multiplication (a.rá). Rearrangement of the rectangle inside this square and "tearing it out" (*nasāḫum*) leaves an excess square, whose side is the deviation of each of the two sides from their average ($\ell-\frac{\ell+W}{2} = \frac{\ell+W}{2}-W = \frac{\ell-W}{2}$). This side (that side which "is equal", íb.si$_8$, along the square area 15´) is 30´. Joining this to one side of the large square gives the length l = 14°30´+30´ = 15; "cutting it off" (*ḫarāṣum*) from the other gives the width W = 14°30´–30´ = 14. Finally, the 2 which were added to the width of the original rectangle (and thus to $\ell+w$) are torn out from W, leaving w = 12. The solution is followed by a proof for control, but even without this the procedure is easily "seen" to be correct once we understand the geometric cut-and-paste operations prescribed by the text.

Next we may look at one of the didactical expositions from Susa, namely TMS IX #1–2[6] – still concerned with a rectangle, whose sides are presupposed to be ℓ = 30, w = 20´ (and the area thus A = 10´):

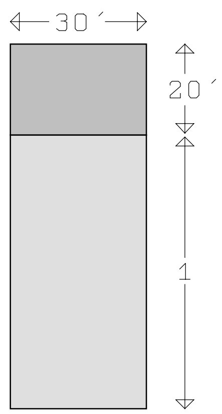

Figure 2. The configuration described in TMS IX #1.

#1

1. The surface and 1 length accumulated, 4[0´. ¿30, the length,? 20´ the width.]

6 Based on the hand copy and transliteration of [TMS, pl. 17, p. 63], with corrections from [von Soden 1964]; I follow my revised text and translation from [Høyrup 2002: 89–91].

a.šà *ù* 1 uš UL.GAR 4[0 ?30 uš? 20 sağ][7]

2. As 1 length to 10´ ⌈the surface, has been appended,]
i-nu-ma 1 uš *a-na* 10 ⌈a.šà daḫ]

3. or 1 (as) base[8] to 20´, [the width, has been appended,]
ú-ul 1 KI.GUB.GUB *a-na* 20 [sağ daḫ]

4. or 1°20´ [¿is posited?] to the width which 40´ together ⌈with the length ¿holds?]
ú-ul 1,20 *a-na* sağ *šà* 40 *it-*⌈*ti* uš ¿NIGIN ğar?]

5. or (that which) 1°20´ toge⟨ther⟩ with 30´ the length hol[ds], 40´ (is) [its] name.
ú-ul 1,20 *it-*⟨*ti*⟩ 30 uš NIG[IN] 40 *šum-*[*šu*]

6. Since so, to 20´ the width, which is said to you,
aš-šum ki-a-am a-na 20 sağ *šà qa-bu-ku*

7. 1 is appended: 1°20´ you see. Out from here
1 daḫ*-ma* 1,20[9] *ta-mar iš-tu an-ni-ki-a-am*

8. you ask. 40´ the surface, 1°20´ the width, the length what?
ta-šà-al 40 a.šà 1,20 sağ uš *mi-nu*

9. [30´ the length. T]hus the procedure.
[30 uš *k*]*i-a-am ne-pé-šum*

#2

10. [Surface, length, and width accu]mulated, 1. By the Akkadian (method).
[a.šà uš *ù* sağ U]L.GAR 1 *i-na ak-ka-di-i*

11. [1 to the length append.] 1 to the width append. Since 1 to the length is appended,
[1 *a-na* uš daḫ] 1 *a-na* sağ daḫ *aš-šum* 1 *a-na* uš daḫ

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[7] This restitution is mine, as are many of those that follow. From the quotation in line 6 the statement can be seen to have given the value of the width; whether the length was also stated explicitly or just presupposed routinely remains a guess, but the reference to the value of the surface in line 2 shows that even the length is supposed to be known.

[8] "Base" translates the logogram KI.GUB.GUB, which is not known from elsewhere (the Late Babylonian value ki.du.du~*kidudûm* is clearly irrelevant). GUB has two different Sumerian interpretations, du / RÁ etc., "to go" [SLa § 268], and gub, "to stand, to erect" [SLa § 267]; to judge from the logographic occurrences, the reduplication is used to indicate iterative and durative aspects. ki may function as a virtual locative verbal prefix, "on the ground" [SLa §306]. A possible reading of the complex thus seems to be ki.gub.gub, "to stand / that which stands erected permanently on the ground".

The reading "coefficient of the length" proposed by Kazuo Muroi [1994] can be safely disregarded, both because it suggests (without collation of the tablet) the reading to be changed into *ki.gub uš, and because the supposedly corroborative evidence in the text BM 15285 is indeed counter-evidence – cf. [Høyrup 1995b].

[9] This follows the hand copy of [TMS], against the transliteration.

12. [1 to the width is app]ended, 1 and 1 make hold, 1 you see.
[1 *a-na* sağ d]aḫ 1 *ù* 1 NIGIN 1 *ta-mar*

13. [1 to the accumulation of length,] width and surface append, 2 you see.
[1 *a-na* UL.GAR uš] sağ *ù* a.šà daḫ 2 *ta-mar*

14. [To 20´ the width, 1 appe]nd, 1°20´. To 30´ the length, 1 append, 1°30´.[10]
[*a-na* 20 sağ 1 da]ḫ† 1,20 *a-na* 30 uš 1 daḫ 1,30

15. [¿Since? a surf]ace, that of 1°20´ the width, that of 1°30´ the length,
[¿*aš-šum*? a.š]à *šà*† 1,20 sağ *šà* 1,30 uš

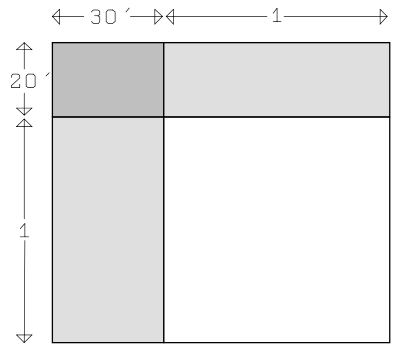

Figure 3. The configuration of TMS IX #2.

16. [¿the length together with? the wi]dth, are made hold, what is its name?
[¿uš *it-ti*? sa]ğ† *šu-ta-ku-lu mi-nu šum-šu*

17. 2 the surface.
2 a.šà

18. Thus the Akkadian (method).
ki-a-am ak-ka-du-ú

Here no problems are solved – what we see are prolegomena to a solution. In #1, the arithmetical sum of area and length is told to be 40´. This time, the length is not silently presupposed to be “broad”, instead a ficticious breadth 1 is introduced (designated KI.GUB.GUB, possibly to be read “base”) – cf. Figure 2. #2 uses the same trick to the case where $A+\ell+w = 1$ is given – cf. Figure 3. Now it is taken for granted that addition of $1\cdot\ell$ corresponds to the introduction of a new width $W = w+1$, and addition of $1\cdot w$ corresponds to the introduction of a new length $L = \ell+1$, with the consequence, however, that a square 1×1 is added. In #3 of the tablet, which solves the problem $A+\ell+w = 1$, $^1/_{17}(3\ell+4w)+w = 30´$, L and W are then spoken of as “the length / width of 2 the surface”. The “Akkadian method” of the text is likely to refer to the trick of the quadratic completion.

The third illustrative example is YBC 6967[11]:

Obv.

1. [The *igib*]*ûm* over the *igûm*, 7 it goes beyond
[igi.b]i *e-li* igi 7 *i-ter*

2. [*igûm*] and *igibûm* what?
[igi] *ù* igi.bi *mi-nu-um*

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[10] My restitutions of lines 14–16 are somewhat tentative, even though the mathematical substance is fairly well established by the parallel in lines 28–31.

[11] Based on the transliteration in [MCT, 129].

3. Yo[u], 7 which the *igibûm*
a[*t-t*]*a* 7 *ša* igi.bi

4. over the *igûm* goes beyond
ugu igi *i-te-ru*

5. to two break: 3°30´;
a-na ši-na ḫe-pé-ma 3,30

6. 3°30´ together with 3°30´
3,30 *it-ti* 3,30

7. make hold: 12°15´.
šu-ta-ki-il-ma 12,15

8. To 12°15´ which comes up for you
a-na 12,15 *ša i-li-kum*

9. [1` the surf]ace append: 1`12°15´.
[1 a.ša$^{!}$]$^{a\text{-}am}$ *ṣí-ib-ma* 1,12,15

10. [The equalside of 1`]12°15´ what? 8°30´.
[íb.si$_8$ 1],12,15 *mi-nu-um* 8,30

11. [8°30´ and] 8°30´, its counterpart, lay down.
[8,30 *ù*] 8,30 *me-ḫe-er-šu i-di-ma*

Rev.

1. 3°30´, the made-hold,
3,30 *ta-ki-il-tam*

2. from one tear out,
i-na iš-te-en ú-su-uḫ

3. to one append.
a-na iš-te-en ṣí-ib

4. The first is 12, the second is 5.
iš-te-en 12 ša-nu-um 5

5. 12 is the *igibûm*, 5 is the *igûm*.
12 igi.bi 5 *i-gu-um*

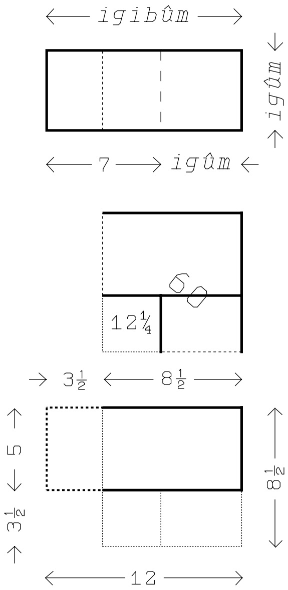

Figure 4. The procedure of YBC 6967.

The problem deals with a pair of numbers belonging together in the so-called table of reciprocals (but since the numbers are 12 and 5 the problem illustrates that this was at least originally a tabulation of aliquot parts of 60, not reciprocals proper, i.e., parts of 1). The numbers are designated *igûm* and *igibûm*, loanwords from the Sumerian meaning "the igi" and "its igi"; as can be seen from the reference in obv. 9 to their product 1` as a "surface" (a.ša), they are represented by the sides of a rectangle with area 1`.

The procedure is similar to what we encountered in AO 8862 #1 (not identical, since the difference between the sides and not their sum is given). At some points, however, the formulations are different. "Breaking" now only stands for the bisection, the construction of the rectangle is a distinct operation (making the sides "hold" each other); the determination of the area, on the other hand, is thought of as automatically implied by the construction, and the numerical computation thus not mentioned. Moreover, in rev. 1–3 we notice that the deviation from the average – corresponding to the part of the rectangle that was broken off and moved around – is "torn out" from one side of the completed square before being "appended" to the other. It is, indeed, *the same piece* which is involved, and it is recognized that it has to be at disposition before it can be added (in all cases where no such constraint is present addition precedes subtraction in Babylonian just as in modern texts).

Originally, the first simple supra-utilitarian problems about rectangular (and square) fields and their sides will have been borrowed by the early Old Babylonian school from a non-scribal ("lay"), presumably Akkadian-speaking surveyors' environment of oral cultural type, among whom a small set of riddles of this kind circulated (and continued to circulate until the Middle Ages, surviving several language changes).[12] AO 8862 is a witness of the early phase of the adoption, TMS IX and YBC 6967 of the later developments that took place within the school environment.

Several characteristic aspects of this development are illustrated by the differences between our three texts. First of all, the terminology of AO 8862 is vacillating – thus the initial construction of the rectangle is referred to as a process of "making hold", whereas slightly later that of the square on $\frac{\ell+W}{2}$ is inherent in the "breaking" of $\ell+W$. Further on in the text (which contains several problems) still other variations are found. We also notice a tendency to "tear out" from surfaces but to "cut off" from linear extensions, but this distinction is not respected absolutely. In later times, the terminology becomes much more uniform; it is not the same everywhere, but most of the corpus falls in groups, each of which follows a very precise canon.[13]

The fate of the "broad lines" is also noteworthy. "Broad lines" are widespread in traditional non-school-based practitioners' traditions, in which the standard width can be assumed to be known by "everybody" – see [Høyrup 1995c]; since they are also assumed in the early Old Babylonian texts we may assume that they had belonged to the practice of the lay surveying environment.[14] Schools and similar institutions, however, tend to be unhappy with this practice, since the tacit conflation of lines and areas impede didactical explication. In the *Laws* (819D–820A, trans. [Bury 1926: II, 105–107]), Plato tells that teachers should "clear away, by lessons in weights and

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[12] The arguments leading to this conclusion are complex and cannot be repeated here. I first presented them in [Høyrup 1995a] and [1996].

[13] These canons are described in detail in [Høyrup 2000].

[14] A parallel is the Babylonian metrology for volumes: since heights and depths are invariably measured in kùš, areas can be considered "thick" and volumes hence measured in the same unit as areas. The use of the term "raising" (*našum*) for the determination of a concrete magnitude by multiplication is almost certainly derived from this practice: the volume of a prism with base A and height h is found by "raising" the virtual height 1 kùš to the real height.

measures, a certain kind of ignorance, both absurd and disgraceful, which is naturally inherent in all men touching lines, surfaces and solids", and make students understand that these categories are "neither absolutely nor moderately commensurable" even though all are measured in feet.

The Old Babylonian school masters coped with this problem in two steps. One was to ensure that problems were not formulated in a way that presupposes that lines can be joined to surfaces, and surfaces to volumes. Instead problem statements came to make use of the "accumulation" addition, a symmetrical addition of measuring numbers. But this transformation (which is found in all later text groups) could only make sense of the statements, and not of procedures which still had to build on the suspicious operations. Here the problem was solved by *representing* explicitly the side to be added by a rectangle with the same length and of breadth 1 – in TMS IX #1 explained as a "base", in BM 13901 termed *wāṣītum* – something which "goes out" or "protrudes" – and in YBC 4714 regarded as a "second width" (with the difference that this width is now the coefficient, not 1); since the usage is changing and the mechanism is not made explicit in other texts we may suppose it to be a later innovation than the change of additive operation.

The procedures used to solve AO 8862 #1 and YBC 6967 may be characterized as "naive", in the sense that they are "seen directly" to be correct but explicit reasons for this correctness are neither given nor asked for. In contrast, proofs like those of *Elements* II.5 and II.6 (analogues of the two Babylonian solutions) are "critical" in the Kantian sense, showing via their appeal to definitions, postulates and axioms *why* and *under which conditions* the proofs hold true. In this sense, already the refusal to join a length to a surface but in particular the introduction of the "base" and its equivalents must be understood as the outcome of a "critique of mensurational reason".

Another instance of critique is the precedence of "tearing-out" over "appending" in the final steps of YBC 6967. This concern for concrete meaningfulness might look as, and has indeed been taken as an expression of a "still concrete" mode of thought unfit for abstraction. It turns out, however, that early texts as well as those later texts whose phraseology betrays vicinity to the lay origins use the single phrase "append and tear out", that is, do not respect concreteness. What we find in YBC 6967 is thus a parallel to what happened in Greek arithmetic when number had to be defined after having been used for millennia by practitioners: it became a "collection of units", with the consequence that both 1 and fractions had to be excluded.

The establishment of a terminological canon is a way in which the mathematical field is submitted to conceptual order and demarcated from general language and practice, and in so far it is a genuine mathematicians' exercise. *Critique*, on its part, comes close to being a distinctive characteristic of ancient Greek mathematics; if Euclid and his predecessors count as mathematicians – which hardly anyone will deny – the modest Old Babylonian commencement of a critical endeavour may be conceived similarly.

Once we acknowledge this, we may return to the supra-utilitarian problems. Are these not instances of "pure mathematics", and isn't pure mathematics another way to demarcate mathematics proper from non-mathematics?

To this we may first object that mathematics as a whole, utilitarian training texts and supra-utilitarian problems together, seems to have constituted a cognitively delimited domain in the Old Babylonian school. Some texts are thematic, and contain problems that can be seen to belong within a particular mathematical field – "algebraic" problems about squares (e.g., BM 13901); "non-algebraic" problems about a subdivided square (BM 15285); "algebraic" problems about prismatic excavations (BM 8200+VAT 6599); utilitarian and "algebraic" problems about the labour costs of prismatic excavations (e.g., YBC 4662); "algebraic" problems about squares and rectangles combined with experiments with composite fractions (e.g., TMS V); etc. Other texts are "anthology texts", combining utilitarian and supra-utilitarian problems dealing with many topics. But apart from school pads carrying a writing exercise on one side and a numerical computation on the other no texts combine mathematics and non-mathematics.

Next we may observe that our dichotomy "pure"/"applied" mathematics is the outcome of a conceptual confusion. Originally (e.g., in Bacon's formulation) "pure" mathematics is opposed to "mixed" mathematics, the former dealing with wholly abstract quantity and number, the latter (Aristotle's "more physical" branches of mathematics) with mathematicized reality. But mixed mathematics may certainly be theoretical and not aimed at practical application[15] – we may think of Euclid's *Optics*, of Ghetaldi's Archimedean proof from 1603 that the concept of density makes sense even if applied to volumes whose ratio is irrational (certainly not anything a practical mechanic would bother about), or of the bulk of articles in *Journal of Mathematical Physics* (at least as I remember them from the 1960s).

Babylonian supra-utilitarian problems are not pure in the original sense, they always deal with real-life entities, with mathematicized reality. Though not applicable in real practice, moreover, they often pretend to deal with practical tasks, and the same theme text will often start with the useful and then pass on to the supra-utilitarian.

In itself, the predominantly supra-utilitarian interest of the texts is thus no reason to regard their authors as "mathematicians". Their aim is not insight, not investigation of principles, the establishment of formal correlations, or anything of the kind. Supra-utilitarian problems are an expression of Old Babylonian "scribal humanism" or nam-lú-ulù, on a par with the reading and speaking of Sumerian: proofs that the scribe is somebody special, able to resolve not only the trite problems that present themselves in scribal everyday but even the most sophisticated ones that might be imagined (by

[15] Wolff [1716: 866*f*], more articulate about the distinction than other writers I know of, speaks of the former as "*mathesis mixta*, die angebrachte Mathematick", of the latter as "*mathesis practica*, die ausübende Mathematick".

other scribes) – cf. [Høyrup 1994].

However, if they are to serve this purpose, the supra-utilitarian problems must be resolvable by methods at hand. For riddles like AO 8862 #1 and YBC 6967, this is easily ascertained by construction backwards from the known end result, once the trick of the quadratic completion is familiar. But what about finding the rates (inverse prices) at which a given quantity of oil is bought and sold if the total profit and the difference between the rates is given (TMS XIII)? Or what about that of finding the sides of a rectangle from its area and from the area of another rectangle whose length is the original diagonal and whose width is the cube constructed on the original length? Both are indeed resolvable, the first leading to the problem of a rectangle for which the area and the difference between the sides is given, the second to a similar problem in which one of the rectangular sides is the square on the square of the original length (TMS XIX). Or what about problems about rectangles in which not only the sides of these but also the coefficients of the equations defining them are asked for (YBC 4713 #1–8)?

It is not impossible to understand how the resolvability of such problems could be predicted by the authors of the texts; I shall omit the argument, but see [Høyrup 2002: 199, 205] for TMS XIX #2 and YBC 4713 #1–8. Yet predicting it requires fairly deep mathematical insight into the structures that are dealt with – considerably more than needed for solving the problems thesmelves. We posses no texts containing the investigations that produced these insights, and they may never have existed as written texts; but the work must have been done, and done systematically: it is extremely unlikely that an eighth-degree problem constructed at random (and TMS XIX is of the eighth degree!) ends up by being resolvable by a cascade of quadratic equations.

The quest for insight *per se* may not have been what moved those who produced the insights; their aim was probably the invention of problems that might serve the display of scribal virtuosity – that of the students or, perhaps more likely, that of the teacher. But whatever the motive, their activity created "insights in the formal properties of mathematical objects", and correlated "the properties of different mathematical objects or classes of objects". These phrases were borrowed from my initial characterization of the activity of the mathematician, and even in this respect it is thus permissible to see at least this group of Old Babylonian mathematical authors as "mathematicians". Since some of their sophisticated inventions circulated widely with no or little change we may presume that most mathematical authors copied or borrowed, understanding how the sophisticated problems should be solved (some texts actually suggest that not everybody understood equally well) but not how it had been originally determined that these striking problems were resolvable. Nor is there any reason to assume that all the mathematical authors engaged in critique or in the standardization of terminologies. "Mathematicians" may have been a small minority among them. But they were present, and if they did not create it they shaped Old Babylonian mathematics in characteristic ways.

Tablets referred to

AO 6770. Published in [MKT II 37*f*; cf. III 62*ff*].
AO 8862. Published in [MKT I 108–113, II Taf. 35–38; III 53].
BM 13901. Published in [Thureau–Dangin 1936].
BM 15285. Published in [MKT I 137]; with an additional fragment in [Saggs 1960]; with yet another in [Robson 1999: 208–217].
BM 85200+VAT 6599. Published in [MKT I 193*ff*, II Pl 7–8 (photo), Pl 39–40 (hand copy)].
IM 43993. Preliminary publication in [Friberg and al-Rawi 1994]. 85 n.111, 322–324, 322 n.368, 338, 343, 372
IM 52301. Published in [Baqir 1950a].
TMS V. Published in [TMS 35–49, Pl 7–10].
TMS VII. Published in [TMS 52–55, Pl 14–15].
TMS IX. Published in [TMS 63*f*, Pl 17].
TMS XIII. Published in [TMS 82, Pl 22].
TMS XVI. Published in [TMS 91*f*, Pl 25].
TMS XIX. Published in [TMS 101, Pl 28*f*].
YBC 4662. Published in [MCT 71*f*, Pl 8].
YBC 4713. Published (with YBC 4668 and YBC 4712) in [MKT I 422–435, III 61*f*, Taf 2].
YBC 4714. Published in [MKT I 487–492, II Taf 60].
YBC 6967. Published in [MCT 129, Pl 17].
YBC 8633. Published in [MCT 53, Pl 4].

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